UNDERSTANDING TRIGONOMETRIC RELATIONSHIPS BY GROUNDING RULES IN A COHERENT CONCEPTUAL STRUCTURE

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Kevin W. Mickey

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Jay McClelland, Primary Adviser

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Michael Frank

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Dan Schwartz

Approved for the Stanford University Committee on Graduate Studies.

Patricia J. Gumport, Vice Provost for Graduate Education

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Abstract

Mathematical cognition relies on a variety of representations, from algebraic rules to diagrams that convey the meaningful relationships expressed in the algebraic rules. My dissertation shows how a diagram that captures the conceptual structure underlying the domain of trigonometry can make symbolic expressions into more meaningful rules that can be applied and generalized successfully.

We presented students with trigonometric identity problems, and examined the representations they used. Students most often reported using the unit circle, a diagram that captures the meaning of trigonometric expressions in an integrated way. Those who reported using the unit circle also tended to have higher accuracy. Other students, who did not report using the unit circle or reported using it less often, were more likely to rely on heuristics and rules that had no connection to the meanings of the trigonometric constructs involved.

We created a formal rule-based lesson and another lesson which instead grounded relationships in the unit circle, and we randomly assigned students to each lesson. Students in the grounded lesson condition showed improved accuracy both on problems taught in the lesson and on problems held out for transfer. While the unit circle could be used simply as a standalone procedure to solve problems, some students appeared to apply an understanding of the relationships and symmetries in the unit circle to extend and combine rules.

Students reported relying slightly more on a rule or formula when solving taught problems than

transfer problems. However, across all problem types, students often reported relying on both the unit circle and a rule or formula on a particular problem. As students gained experience solving problems after the grounded lesson, they tended to reduce their active use of the unit circle, as indicated by their self-reported strategies and by their interactions with an external unit circle tool.

Across our studies, the unit circle revealed itself to be a strong, coherent conceptual structure, and it creates a fertile ground for learning and understanding trigonometry through the interplay of visuospatial and rule-based approaches. Students learn how to map parts of a symbolic expression onto meaningful properties of the unit circle, and this grounded understanding facilitates application and generalization of rules in order to solve problems successfully. Helping students from all backgrounds master these grounded rules remains a challenge for teachers and cognitive psychologists.

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Chapter 1

Introduction

There is a fundamental contradiction at the heart of mathematics. On the one hand, its concepts can be immaterial, amodal, and incredibly abstract, and its proofs seem to be deduced from entirely a priori premises. Textbooks include tables of algebraic identities, which can appear (both figuratively and sometimes literally) all Greek to students at first, but the beauty and expressive power of mathematics emerges as one gains facility in the interpretation and use of these expressions to support reasoning. On the other hand, mathematics is indispensable to scientists who use empirical methods to describe and explain the tangible, physical world. Visuospatial representations seem to play a crucial role in mathematics, from discovery to formal proofs to communication and explanation. Faced with this apparent paradox, what is the true nature of mathematical knowledge? Although I by no means have a complete answer, this exploration of trigonometric rules grounded in a visuospatial conceptual structure will inform both how cognitive scientists think about the nature of mathematical knowledge and how educators might help students to develop a productive understanding of mathematics.

A growing body of evidence suggests that we may have an intuitive number sense and that visuospatial processes may ground simple numerical and arithmetic reasoning (Dehaene, 1997; Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999; Gallistel & Gelman, 2005). Yet higher level mathematical cognition is often assumed to consist of the manipulation of structured arrangements of symbols according to rules sensitive to the structure but not the content of the symbols (Harnad, 1990). Mathematicians and philosophers like Russell, who proclaimed that "all Mathematics is Symbolic Logic" (1903), attempted to formalize mathematics as an extension of logic, and soon after the introduction of electronic computers, symbolic rule-following programs were developed to prove logic theorems (Newell, Shaw, & Simon, 1958) and to solve mathematical equations of unbounded complexity (Martin & Fateman, 1971). Furthermore, some have argued that symbolic rule-following is essential for many types of systematic cognition (Fodor, 1975; Marcus, 2003). However, the laws of logic may not be the laws of thought.

Indeed, mathematical thinking often relies on visuospatial reasoning (Norma C Presmeg, 2006). Many people may agree that visuospatial reasoning and rule-based reasoning are both necessary to learn mathematics successfully, but their roles are sometimes portrayed as independent and ultimately siloed from each other. What if visuospatial reasoning actually facilitates rule-based reasoning? Such facilitation could occur through what Case et al. (1996) calls a core conceptual structure, which links different representations to support understanding. What if our representations of mathematical relationships have both visuospatial features and rule-like properties? While some laboratory scientists might isolate these reasoning styles and pit them against each other as competing experimental conditions, I will explore the nature of mental representations using the same synergies and mutual support that some teachers emphasize and that many mathematicians exploit. Trigonometry, a branch of mathematics often taught in pre-calculus courses, sits at the intersection of algebra and geometry, and it offers rich opportunities for investigating these questions about mental representation in productive mathematical thinking. Before we examine how visuospatial reasoning and rule-based reasoning might work together, though, it might be worth asking what I mean by visuospatial reasoning, and what I mean by rules.

1.1 Visuospatial Reasoning

I use the term "visuospatial" to refer to spatial representations that are most commonly depicted visually. The visual modality is not necessary or intrinsically important – in fact, blind mathematicians report relying on spatial representations (Jackson, 2002). However, typical spatial representations are diagrams and graphs that we perceive (or imagine) visually. As a result of this fact and as a result of our immense history of visual experiences, spatial representations may be influenced or constrained in part by visual expertise. Although visuospatial representations are often considered "concrete" due to their spatial extent in the physical world, they can be quite abstract in other ways, lacking perceptual detail or narrative context. Previous research has distinguished between schematic and pictorial representations (Hegarty & Kozhevnikov, 1999) or spatial and object imagery (Kozhevnikov, Kosslyn, & Shephard, 2005). My usage corresponds to the schematic or spatial notion of visuospatial representations. Others have discussed types of visuospatial reasoning in the domain of mathematics. Norma C. Presmeg (1986) classified images used by students into five categories:

1. concrete, pictorial imagery,
2. pattern imagery,
3. memory images of formulae,
4. kinaesthetic imagery, and
5. dynamic imagery.

I use "visuospatial reasoning" to refer to any of three types (or any combination): pattern imagery, kinaesthetic imagery, and dynamic imagery. Although potentially co-occurring with these processes, the first category, concrete pictorial imagery, aligns more with the vivid pictorial or object-based imagery discussed above. Memory images of formulae seem to miss my intended meaning of "visuospatial reasoning." Manipulation of symbolic expressions has indeed been shown to involve significant use of perceptual and motor systems, whether during written production (Landy & Goldstone, 2007a), judgments of validity (Landy & Goldstone, 2007b), or learning (Ottmar, Landy, & Goldstone, 2012). However, symbol manipulation has strikingly different affordances and different constraints from prior knowledge and experience than visuospatial representations. I consider visuospatial manipulation of symbolic expressions to be an alternative method to using formal reasoning, where both methods operate on the same representations.

1.2 Rules

Rules may be difficult to define, but I will discuss some properties of rules and rule-based thinking. Brown (2008) identifies several conditions that seem necessary in order to describe someone as using or following a rule. First, he distinguishes a rule from an order: someone typically follows an order once at a particular moment in a particular situation, whereas a rule must cover an indefinite number of situations. At the same time, rules must be graspable by a finite mind (and potentially capturable in a finite description), despite covering infinite instances. Finally, using or following a rule means that it guides or has causal effect on our behavior. Note that our behavior can often be described by a rule without it playing any causal rule in one's mind.

This final idea, that we may behave in accordance with a rule without actually using a rule, has been a recurring theme in cognitive science, in particular in connectionist and emergentist perspectives (e.g., Rumelhart & McClelland, 1986). McClelland (1989) presented a model of learning about the role of weight and distance from a fulcrum in balance-scale problems. The network learned from the configuration of weights and their locations on the scale to predict whether the scale would balance or whether the left or right side would go down. The model demonstrated good conformity to the stage-like developmental progression seen in children and previously characterized by rules (R. S. Siegler, 1976), even though the underlying learning from each new experience was gradual and continuous. Similarly, in mathematical cognition, rule-based manipulation of structured expressions is often assumed (Anderson & Lebiere, 1998) but may not always be necessary. As students learn to solve addition problems, they may see $3 + 4 + 9 = 9 + \_$ and answer 25. This add-all strategy could be the result of applying simple rules, before learning more complex rules of manipulating arithmetic syntax. However, domain-general principles without explicit representation of strategies or rules could also explain this behavior. Mickey & McClelland (2014) use a parallel distributed processing approach to model the change resistance (McNeil & Alibali, 2005) that emerges from prior experience. To avoid some of these difficulties with implicit rules, I will be focusing largely on explicit rules, which are prevalent in mathematics. While questions about defining rules and identifying rule use may remain, this work primarily explores the relationship between rules and visuospatial reasoning in the context of mathematics.

1.3 Grounding Rules in a Visuospatial Conceptual Structure

Visuospatial reasoning is surprisingly powerful for understanding or appreciating mathematical relationships. This surprise stems from the contradiction introduced in the first paragraph of this document, that is, the contrast between formal rules and visuospatial representations. Some may claim that the goal of mathematics is to identify explicit relationships through valid formal reasoning. In fact, the Oxford English Dictionary states that mathematics refers, “in a strict sense, to the abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations . . . ” (“Mathematics, n. pl.” 1989). Departing from this definition (while also focusing on the cognitive activity rather than the social artifact), I believe that our conceptions of spatial and numerical relations can grow to be quite advanced, through the accumulation of knowledge both over our lifetime and over our culture’s intellectual history. Ultimately, we can make sound mathematical arguments that invoke formal rules by grounding those rules in a visuospatial conceptual structure.

Without grounding, without any connection between symbols and their referents, symbols are “amodal, abstract and arbitrary” (Glenberg, Gutierrez, Levin, Japuntich, & Kaschak, 2004). Reasoning with symbols in such an empty manner is usually difficult, as demonstrated in Wason’s (1966) selection task, when not informed by life experience or guided by pragmatic goals (Cheng & Holyoak, 1985). Glenberg et al. (2004) argue that we understand the meaning of words through the simulation of their content. According to their indexing hypothesis, we index words and phrases to objects in the environment, and then produce a simulation that is constrained and guided by affordances of the objects. Bransford, Barclay, & Franks (1972) argue that constructing such a simulation is what underlies our interpretation of sentences and our memory of their semantic content.

When it comes to grounding mathematical expressions, we confront issues regarding the referent of a mathematical symbol. For instance, structuralists like Resnik (1981) argue that mathematical objects do not have any meaningful content per se, because mathematics is simply the study of structure:

> “In mathematics, I claim, we do not have objects with an ‘internal’ composition arranged in structures, we only have structures. The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are structureless points or positions in structures. As positions in structures, they have no identity or features outside of a structure.” (Resnik, 1981, p. 530)

This issue may not actually distinguish mathematics from language. While some words do refer to specific objects, there are also words whose referents are object categories, events, actions, and abstract relations. In mathematics, we are often referring to quantities such as length or area that are measurable properties of real or imagined objects (A. G. Thompson, Philipp, Thompson, & Boyd, 1994). While mathematics allows more opportunities for purely formal symbol manipulation, mathematics educators have argued persuasively that such actions are highly error prone. Instead, maintaining contact with the referenced context deepens engagement with the underlying relationships (A. G. Thompson et al., 1994), helps prevent errors (Mayer & Hegarty, 1996), and facilitates transfer (Lewis, 1989).

These referent quantities are not isolated properties of a single representation in a student's mind. Rather, mathematics relies on what Case et al. (1996) call core conceptual structures. He defines these structures as "networks of semantic nodes and relations that represent children's core knowledge in a domain and that can be applied to the full range of tasks that the domain entails" (Case et al., 1996, p. v). Although not restricted to mathematics, Case et al. (1996) considered the development of the number concept, from a counting schema unrelated to cardinality, to a unified mental number line and later relational reasoning. As we read a symbolic expression, we map those symbols to components of a visuospatial conceptual structure. We can then perform certain procedures (or simulations) within certain constraints to identify equivalent quantities and other relationships. Formal expressions are no longer meaningless, but rather enriched and defined in terms of the conceptual structure.

1.4 Grounding Trigonometric Rules in the Unit Circle

Trigonometry is an exciting domain for cognitive scientists because it allows for a wide range of spatial and symbolic representations. It also can be very challenging for students, and so presents a real opportunity for educational advances. Finding better ways of teaching trigonometry may be an important way to remove some of the barriers to college and STEM careers. In this line of research, I am investigating the role of visuospatial representations and algebraic rules and their interaction in learning to solve trigonometric identities. In particular, I am interested in the nature of rules that have been grounded in a visuospatial conceptual structure called the unit circle, shown in Figure 1.1.

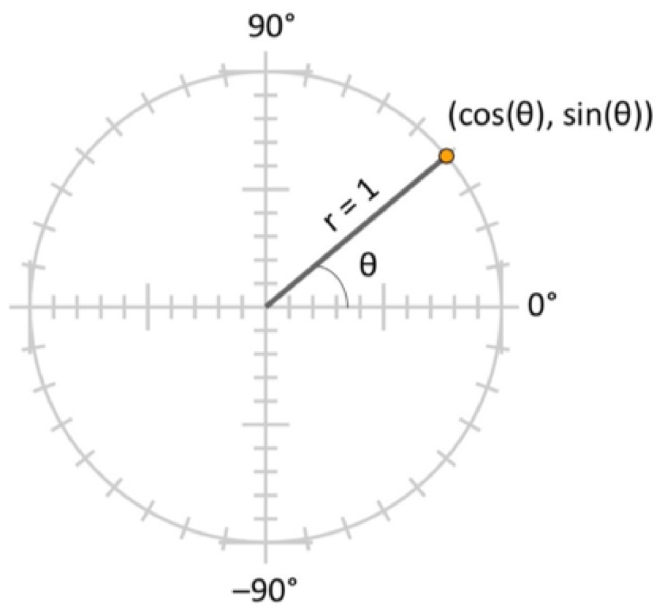


Figure 1.1: The unit circle.

The unit circle has a radius of 1 unit, and a center at (0, 0) on the (x, y) coordinate plane. Angles can be visualized on the unit circle by defining the initial side on the positive side of the x-axis, and by rotating up (counter-clockwise) for positive angles and down (clockwise) for negative angles. A complete circle is 360 degrees or 2π in radians. The cosine of an angle is equal to the x-coordinate of the point where the angle's terminal side intersects the circumference of the circle. Likewise, the sine of an angle is equal to the y-coordinate. (Note that with a radius of 1, trigonometric ratios whose denominator is the hypotenuse of a triangle can be simplified when the unit radius is the hypotenuse.)

The unit circle is a core conceptual structure in trigonometry that integrates schemas for the X-Y coordinate plane and for angles of a circle, as illustrated in Figure 1.2. Lakoff & Nunez (2000) identified the unit circle as an example of a conceptual (and perhaps metaphorical) blending in mathematics. Fauconnier & Turner (1998) (who use imaginary numbers as another example of conceptual blends) argued that productive insights can emerge from conceptual integration through both composition and completion of concepts. I will argue that students do indeed benefit from the unit circle's fundamental composition of simpler concepts, as well as other advantageous properties of a core conceptual structure.

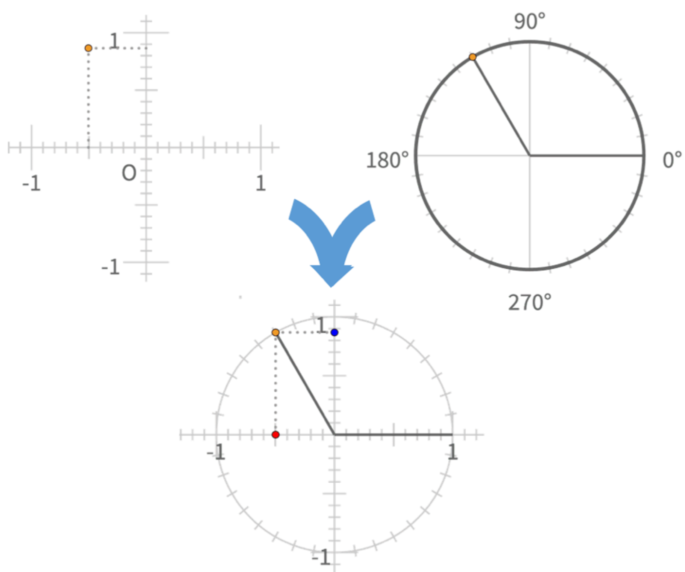


Figure 1.2: The unit circle is a core conceptual structure in trigonometry that integrates schemas for the X-Y coordinate plane and for angles of a circle.

Chapter 2 introduces Studies 1 and 2, in which Stanford undergraduates saw trigonometric expressions (e.g., $\sin(-70+180)$) and tried to identify which of four simpler expressions was equivalent to the given expression. We tested three different approaches to learning trigonometry: a brief lesson presenting trigonometric identities as purely formal rules, a brief lesson grounding these relationships in the unit circle, or no lesson, as a baseline. Students solved one block of problems, saw either the formal lesson or the grounded lesson or no lesson, and then solved another block of problems. As expected, both lesson conditions showed significantly more improvement from pre- to post-test than participants who received no lesson. Interestingly, the formal lesson led only to improvements on problem types that were included in the lesson, whereas the grounded lesson caused improvements on not only taught problem types but also problem types that were held out of the lesson to assess transfer.

1.5 Mechanisms of Transfer

While knowledge of mathematical relationships is necessary for successful problem solving, students must ultimately be able to apply their knowledge in flexible ways to novel problems. Chapter 3 explores how visuospatial reasoning and rule-based reasoning work together to extract some sort of meaning out of novel expressions and to utilize that understanding in novel problems.

Gestalt psychologists have long emphasized the role of productive insights. For instance, if the child knows how to calculate the area of a rectangle and appreciates the meaning of area, such a child may find the area of a parallelogram without having memorized any dedicated formula (Wertheimer, 1959). The area of a parallelogram can be found by decomposing it into its parts (a rectangle with a triangle on each end) and by recognizing that one of the triangles can be moved to the other side to construct a rectangle of the same area as the parallelogram. Although initially not knowing (or having forgotten) the formula for the area of a parallelogram, such a child would not only perform at the level of a child who had memorized the formula, but the former child would have demonstrated an understanding of area and its invariance under transformations that the latter child may not have.

We achieve this kind of productive insight by finding a representation that affords reconfiguration – in the case of trigonometry, perhaps by rearranging parts of a symbolic expression and substituting a known identity, or perhaps by aligning a symbolic expression with components of the unit circle.

From a formal, algebraic point of view, the set of rules provided in the formal lesson together with principles of ordinary arithmetic were sufficient to solve all of the identity problems, although in some cases, more than one rule had to be applied to obtain the correct answer. While these steps may seem to belong in an introductory algebra course, the lack of transfer by students in the formal lesson suggests that it may be difficult to recall the appropriate rule, hold it in working memory, and perform algebraic manipulations.

Unlike the formal lesson, the grounded lesson did help students solve transfer problems. There are a number of possible mechanisms through which this could have occurred, but I will focus on three of these mechanisms as potential explanations. Note that these mechanisms may not be mutually exclusive – some combination of mechanisms could describe behavior, both between students and within the same individual.

1.5.1 Unit circle as a problem solving procedure

First, grounding symbolic expressions in a meaningful domain may support flexible reasoning simply as a result of broader facility in that grounded domain. A grounded domain can help us make meaning by facilitating specific inferences about relationships or procedures within the target domain (Kieras & Bovair, 1984). In trigonometry, students can perform a sequence of operations with the unit circle in order to find an answer to a particular problem. Note that there are many possible procedures involving the unit circle, but I will describe one specific instance, as an example. A student looks at the probe expression, locates the angle on the unit circle, and reads the value off of the unit circle, either as a number or as the approximate (directed) length of a line segment.

Next, the student performs the same procedure for the angle in each of the expressions presented as potentially equivalent to the probe expression, and compares the original value to each alternative. In some cases, this comparison may require comparing line lengths on different axes. This procedure can identify equivalent expressions for each type of problem that we presented in our studies, when performed correctly.

One question concerning this kind of procedure is its role in identifying and justifying generalized relationships. For instance, in Study 3, we included θ instead of specific angles in expressions. While rules inherently apply to an indefinite number of instances, the unit circle is a spatial model, which means it represents quantities as specific instances of positions, lines, or angles. Schwartz & Heiser (2006) identify this determinate nature as a feature of spatial representations that can prevent ambiguity and potentially facilitate memory. Mani & Johnson-Laird (1982) in fact tested whether a determinate spatial representation has such benefits. Their participants saw spatial descriptions that were sometimes determinate (the box is to the left of the tree) and sometimes indeterminate (the box is beside the tree), followed by a potential diagram. They were then presented with a recognition test, which included the original sentence, a paraphrased or inferable spatial description, and two confusion items. Subjects correctly rejected the confusion items more often for determinate stimuli than for indeterminate stimuli, but they were slightly worse at distinguishing the actual original from the paraphrased or inferable alternative. Mani & Johnson-Laird (1982) concluded that the determinate relations were more likely being recalled with a mental image or model, which facilitates recognition of spatial relationships, but does not contain the verbatim details of the original sentence. In mathematics, such verbatim details (like ordering of a commutative operation) are superficial. A determinate spatial representation therefore can effectively capture important mathematical relationships, although it may be only a specific instance of a general (indeterminate) relationship. For instance, we may reason about the cosine function by imagining a particular angle, and projecting it onto the x axis. A relationship like $\cos(-\theta) = \cos(\theta)$ that truly abstracts across angle seems impossible to illustrate on the unit circle. There may be some spatial representations that are indeterminate. Consider, for instance, the Necker cube, which has ambiguous depth cues. Perhaps it is interesting precisely because determinate spatial relations are such an expected, core feature of diagrams.

A procedure that involves using determinate spatial representations may, however, involve abstraction across the determinate quantity. If we assume the unit circle requires a specific angle, we still have flexibility in our choice of the angle. The angle instantiated in the unit circle model may not match the angle described in the expression of a problem. For instance, in our trigonometry identity problems, we could instruct students to ignore the angle in a particular expression and to visualize instead a small fixed angle for every problem. This would save time by reducing or even eliminating the effort used to locate an angle and to construct a mental image of that angle. If the angle is within the first quadrant, it may be a prototypical angle that has additional fluency and attractiveness as a prototype (Winkielman, Halberstadt, Fazendeiro, & Catty, 2016). While this strategy may have speed benefits, it fails in two respects. First, visualizing an angle different from the angle used in the probe expression of the problem loses the specific insight and actual applicability to the problem. Second, there is no prima facie guarantee that the properties of the visualized angle extend to the probe angle. Instead of a small fixed angle, we could instruct students to visualize a random angle. This procedure would ensure that nothing is special about the visualized angle in the long run across all students, but it would still be vulnerable on a single try.

1.5.2 Rules and the unit circle as independent strategies

A second possible explanation is that students were utilizing both the unit circle and rules as independent strategies. Visuospatial reasoning can provide students with an effective standalone procedure for deriving answers to problems, but rules may have certain advantages. For instance, rules may be quicker to apply, but harder to transfer (Schwartz & Black, 1996). If this were true, we would expect students to use rules on taught problems, and to use the unit circle on problems for which they cannot recall a rule. Rules also seem better suited for a student to identify and justify generalized relationships involving θ.

1.5.3 Rules grounded in the unit circle

Finally, while students may benefit from having rules as an additional independent strategy, the unit circle and rules may be more intimately related, through mutual support or interaction. The grounded lesson could change the way students discover, use, and remember rules or rule-like representations to solve problems. For instance, in a kind of progressive formalization (Nathan, 2012), the symbolic rule $\sin(x + 180) = -\sin(x)$ may be verbally encoded as "when an angle is rotated halfway around the circle, the y-coordinate of its endpoint has the same value but the opposite sign". This is what I consider a "grounded rule," where the mental representation of an explicit rule is intrinsically tied to a visuospatial conceptual structure (here, the unit circle).

In any visualization with the unit circle, we must be able to quickly perceive equivalent lengths or equivalent angles, and in dynamic visualizations, we must be able to coordinate both the movement of the angles and the projection of those angles onto the appropriate axes. I propose that recognition of symmetries gives us this ability to imagine exact congruence. External representations of platonic shapes are never perfectly exact, nor are their relationships to each other ever exactly congruent. However, we may have a bias toward symmetry that corrects our internal representations, and that allows us to use these representations for mathematical inferences that demand exact congruence. Freyd & Tversky (1984) showed people two-dimensional shapes that were somewhat symmetric, and asked them to compare these probe shapes to more or less symmetric shapes. Participants consistently rated the probe shape more similar to the more symmetric shape, and they were more likely to confuse it with the more symmetric shape, as evidenced by the longer time taken to distinguish the probe shape from the more symmetric shape in a shape matching task. This systematic bias towards more symmetry in our visuospatial representations can also be shown when people attempt to draw a curve on a graph or a river on a map from memory, and their reproductions are more symmetric than the original (subject to the conceptual context of a graph or a map) (Tversky & Schiano, 1989). More generally, we seem to have a strong ability to perceive or imagine the invariance of objects under transformations. For instance, Shepard (2008) highlighted a proof of the Pythagorean Theorem that relies on our intuitive understanding of the invariance of area under rigid transformation.

While we have powerful intuition of invariant relations and symmetries, not all symmetries are created equal. Recognizing identity (if we consider this a symmetry) is quicker and more accurate than recognition of vertical or horizontal symmetry, and recognizing vertical or horizontal symmetry is quicker and more accurate than oblique symmetry (Wagemans, 1997). Children also learn to discriminate and reproduce vertically symmetric patterns first, then horizontally symmetric patterns, and latest in development, patterns with oblique symmetry (Bornstein & Stiles-Davis, 1984). If symmetry mediates the effect of the unit circle on performance, then the unit circle should improve performance on problems with delta equal to 90 (which have oblique or rotational symmetry), but its effect should be larger for problems with delta equal to 180 (which have horizontal and vertical or rotational symmetry), and its effect should be even larger for problems with delta equal to 0 (which have identical symmetry). These predictions correspond with the average accuracy of students on our identity problems.

As an alternative or in addition to symmetries, a student could learn grounded rules by iteratively imagining several random angles. As the number of visualized angles grows, the student gains confidence that the properties observed apply to all angles. A guarantee would only come after an infinite number of angles have been visualized, however. Furthermore, there are time constraints which limit a student to two or three angles, if students even allot time for imagining multiple angles. A low number of samples can succeed in various cognitive domains under basic cost-benefit assumptions (Vul, Goodman, Griffiths, & Tenenbaum, 2014). However, mathematical validity may be a much higher bar than most daily tasks. (Note that it is possible mathematical understanding does not require a demonstration of validity to oneself, but I would argue that an understanding which relies on the authority of a teacher or textbook to generalize a relationship may be insufficient.) What role does inductive reasoning play in a domain rooted in the practice of formal logic? Nisbett, Krantz, Jepson, & Kunda (1983) studied factors that influence whether people use statistical inductive reasoning or some other strategy. While mathematics is well suited on some factors such as the clarity of the sample space and the repeatability of sampling, one factor that potentially discourages inductive reasoning in mathematics is cultural prescription. Mathematicians rely heavily on deductive reasoning, and math educators strive to develop such a capability in their students. Ultimately, students may benefit from the inductive process that I have described when linked with explicitly taught rules, as a path toward grounded rules.

1.6 Readiness for Grounded Rules

Grounding mathematical expressions requires a meaningful referent, and that meaning emerges from a student's existing knowledge. As a result, students who lack relevant prior knowledge may struggle to learn how to apply a grounded conceptual structure like the unit circle. In the first part of Chapter 4, I will consider this effect of prior knowledge, by analyzing its relationship with problem solving after the lesson, and by considering students with a diverse range of prior knowledge, from high school and community college.

By readiness for grounded rules, I also include students who, as they gain experience with the unit circle, may not need to be actively using it in order to succeed at solving trigonometric identity problems. Schwartz & Black (1996) studied the transition from mentally simulating gears turning to a heuristic using the even or odd number of teeth on gears. This process began with fading parts of the gears, then codifying (or discrete labeling with words or gestures), and finally quantitative (or symbolic) casting. They couldn't observe these intermediate representations in their original grad student population, so they had two ways of increasing use. First, they used a population that has less experience / facility with numbers and identifying certain patterns. Second, they encouraged a simplification (like the parity rule), then removed its advantage later, in an attempt to put the participant on the fence between their rules and models. In the second part of Chapter 4, I will explore how students reduce their active use of the unit circle, and whether they are able to maintain successful problem solving, potentially continuing to keep the unit circle as an implicit conceptual foundation of grounded rules.

Chapter 2

The Unit Circle: A Popular and Successful Model in Trigonometry

To explore the nature of mathematical thinking, and in particular the relationship between visuospatial reasoning and rule-based reasoning, I chose to study trigonometry, a domain at the intersection of algebra and geometry. This line of research is important, given the difficulty so many students have in mastering this subject – a difficulty we will document in the study we report here – and given that trigonometry is an essential part of the preparation of students for calculus and therefore for entry into the STEM disciplines. Despite the importance of this issue, we know of little relevant existing research. Other researchers have assumed that visuospatial representations play an important role, using this as a starting place for the development of teaching methods (Kendal & Stacey, 1996; Moore, 2013; C. Ninness et al., 2006, 2009). Visuospatial representations are introduced in trigonometry textbooks, but there is wide variation in the reliance on these representations, with many authors leaning heavily on manipulation of algebraic expressions rather than reasoning directly with visuospatial representations (Foerster, 1990; Hornsby, Lial, & Rockswold, 2010). Thus, our work may help direct future efforts to explore which approaches to emphasize in the teaching of trigonometry.

2.1 General Methods and Preliminary Study

In this study, we asked what kinds of mental representations college students use to solve simple trigonometric identity propblems. An example problem of the type that we have used in our studies is shown in Figure 2.1. Participants saw a probe expression involving the sine or cosine function at the top of a display, and had to indicate which of the four alternative expressions below it had the same value as the probe. These problems can be solved using symbolic rules, spatial representations, and other strategies. We sought to determine what representations participants used, and which ones were associated with better performance.

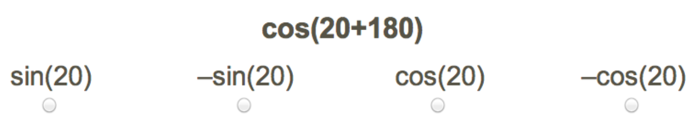


Figure 2.1: An example problem. Participants saw a probe expression at the top of an on-screen display and were instructed to choose the equivalent expression from the alternatives below it.

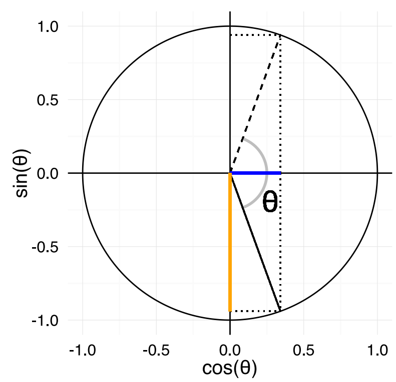
Figure 2.2 shows some of the representations that can be used in solving our problems. As one alternative, trigonometric functions such as sine and cosine can be treated purely symbolically, since these functions have equivalence relationships that can be characterized by such symbolic expressions. In addition, such expressions can be manipulated using standard algebraic rules (e.g., addends may be commuted, the same expression may be added to or subtracted from both sides of an equation, $x + 0 = x$, etc.). The simple trigonometric rules shown, together with basic algebra, are sufficient to solve all of our problems. Alternatively, trigonometric expressions can be defined in terms of geometric objects (lines and angles), their properties (measureable quantities such as lengths and angular measure), and their relationships, and these may be captured in spatial representations. We considered three different spatial representations. One representation sufficient to solve all the problems is provided by visualizing angles as associated with points on the unit circle, and visualizing their sines and cosines as signed distances from the center of the circle to the projections of such points onto the vertical and horizontal axes through the circle's center. The problems can also be solved by visualizing the sine and cosine functions as waves, with the value of an expression corresponding to the vertical position of the appropriate point on its wave function. The relationships needed to solve a subset of the problems can also be understood in terms of the angles and sides of a right triangle. Participants might also rely on mnemonics, such as 'All Students Take Calculus' shown in Figure 2.2, which may draw on both symbolic rules and spatial representations, linked through memorable phrases.

Figure 2.3 shows how several example problems may be solved with trigonometric and algebraic rules (see caption for details). It is worthwhile to note how simple the rule-based derivations are: two require only a single application of a trigonometry-specific rule, while the third requires one general algebraic rule ($x + 0 = x$), and one trigonometry-specific rule. Some of the problems we used require more rule applications. The figure also shows how identities can be established using the unit circle, treating angles as corresponding to positions on the unit circle, and their sines and cosines as corresponding to the projections of these points on the x and y axes through the center of the circle (see caption). Similar visuospatial relationships exist for the waves and (for some problems) for the right triangle.

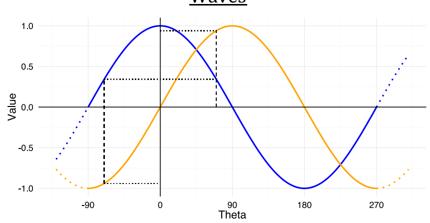
To explore the representations participants used in solving these trigonometric identity problems, we tested 37 undergraduate students at Stanford university. Our preliminary study was intentionally exploratory in nature, and allowed us to develop specific hypotheses to be tested in Study 1. Both

Visuospatial Representations

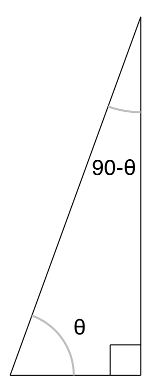
Unit Circle



Waves



Triangle



Symbolic Rules

Trigonometric Identities

sin(−θ) = −sin(θ)
cos(−θ) = cos(θ)
func(θ±180) = −func(θ)
func(90−θ) = opp(θ)

Principles of Algebra

x ± 0 = x
−(−x) = x
−x−y = −(x+y)
x+y = y+x

A Mnemonic

Symbolic Rule Component

Mnemonic	Positive functions	In quadrant:
All	**A**ll	I
Students	**S**ine	II
Take	**T**angent	III
Calculus	**C**osine	IV

Visuospatial Component

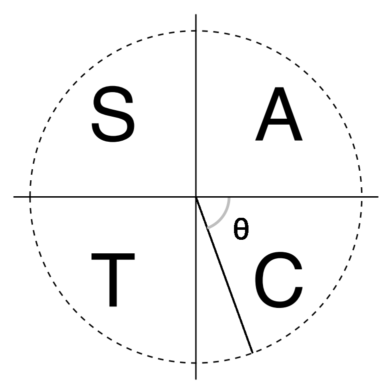
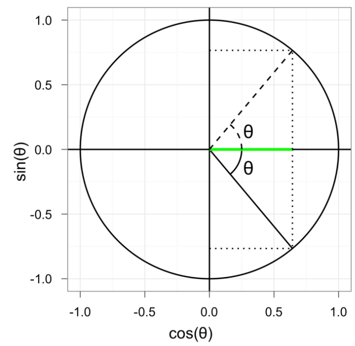
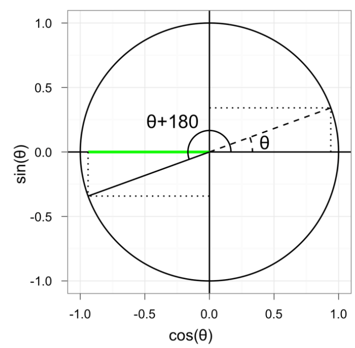


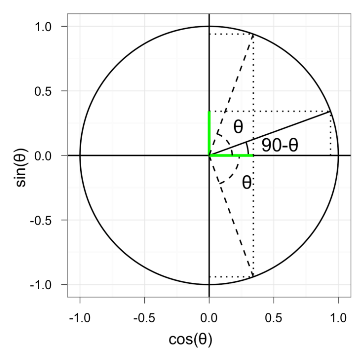
Figure 2.2: Some representations used in the domain of trigonometry. In the rules shown, func could be sin or cos; when func refers to one of these, opp refers to the other.



Expression	Justification
cos(−50+0)	[Problem]
cos(−50)	func(θ±0) = func(θ)
cos(50)	cos(−θ) = cos(θ)



Expression	Justification
cos(20+180)	[Problem]
−cos(20)	func(θ±180) = −func(θ)



Expression	Justification
sin(90−70)	[Problem]
cos(70)	func(90−θ) = opp(θ)

Figure 2.3: Solving trigonometry problems like those in Studies 1 and 2 where a complex expression must be reduced to an equivalent simpler expression, with two procedures shown: the unit circle (left) and algebraic manipulation using structure-sensitive rules (right). On the unit circle (left), an angle is represented by a counter-clockwise rotation of a radial line of unit length, starting from a position aligned with the positive x-axis. A negative angle is represented by a clockwise rotation. The cosine of an angle is the x-coordinate of the endpoint of the radial line where it intersects the unit circle, and also corresponds to the signed length of the projection of the endpoint of the radial line onto the x-axis. Likewise, the sine of an angle corresponds to the y-coordinate of the radial line's endpoint.

studies explored the relationship between self-reported use of alternative representations and success in solving trigonometric identity problems. Participants first solved a set of 40 problems, then answered a few short questions and took a break, then completed forty more such problems, followed by a fuller set of questions about the representations they used and their trigonometry background. During their break, some participants saw a formal rule-based lesson, intended as a brief lesson, some participants saw a lesson grounded in the wave graphs of the sine and cosine functions, and some participants saw no lesson.

In the preliminary study, we found a wide range of variation in performance, with many scoring below 50% correct overall, even though all of the participants had some prior exposure to trigonometry and strong enough quantitative backgrounds to gain admission to the university. We did not find any difference between lessons in their effect on improvement from block 1 to block 2, but this study was limited with its small sample size per group and with a randomly assigned but unfortunately uneven distribution of prior knowledge among lesson conditions. Strikingly, participants performed especially poorly on a problem that taps a basic and explicitly instructed aspect of pre-calculus trigonometry, namely the value of the cosine of a negative argument. Given the probe $\cos(-\theta + 0)$, many participants chose $-\cos(\theta)$, whereas the correct answer is $\cos(\theta)$ (as shown in the top row of panels in Figure 2.3).

Our preliminary study also provided evidence relevant to the role of visuospatial representations in trigonometry. Specifically, we found that one of the visuospatial representations – the unit circle – tended to be favored relative to others, and was also associated with successful performance. Participants reported using the unit circle more than any other representation; self-reported circle use was more strongly associated with greater overall accuracy than other representations, even after controlling for self-reported number of courses involving trigonometry and years since last use of trigonometry; and those who reported always using the unit circle did much better on $\cos(-\theta + 0)$

problems than other participants.

These findings, if reliable, clearly support a role for the unit circle in solving the trigonometric identity problems used in our study and point to a specific role for this representation in allowing mastery of a very basic aspect of trigonometry. Since these findings emerged from an exploratory analysis, we concluded that they should be treated as tentative, subject to confirmation in a replication of the preliminary study. Therefore, we conducted Study 1, with a larger group of participants, all with no lesson, and some methodological refinements, aiming to determine whether the same patterns emerged.

2.2 Study 1: Exploring Trigonometric Reasoning

2.2.1 Method

Participants

We recruited 50 Stanford University undergraduate students to participate in a single hour-long session in exchange either for credit in an introductory psychology class or for pay ($12). We decided a priori to stop collecting data once we reached 50 participants. We chose a round number somewhat greater than the number of participants in the pilot study to allow us to achieve somewhat tighter confidence intervals for our dependent measures.

Materials

The problems used each consisted of a probe and four choice alternatives (Figure 2.1). The probe was always an expression of the following general form:

$$\text{func}(\pm\theta \pm \Delta)$$

where func was either sin (sine) or cos (cosine), and θ and Δ were numeric expressions in degrees.

The value of θ was drawn from the set {10, 20, 30, 40, 50, 60, 70, 80} and could be positive or negative while $\pm\Delta$ could take any of the values (–180, –90, +0, +90, +180). The order of the terms within the parentheses of the probe varied so that θ could occur first or second. If the first term was positive, its sign was not displayed. The choice alternatives were always of the form:

$$\sin(\theta) \text{ OR } -\sin(\theta) \text{ OR } \cos(\theta) \text{ OR } -\cos(\theta)$$

where the value of θ was the same as its value in the probe. For each participant, two blocks of forty problems were generated according to a 2x2x5x2 design in which function (sin or cos), sign of θ (positive or negative), value of $\pm\Delta$, and order (θ before Δ or Δ before θ) were fully crossed. The value of θ was selected randomly and independently on each trial.

Procedure

Each participant was tested individually while seated in a quiet laboratory room. At the beginning of each block, participants read instructions indicating that their task was to consider the expression at the top of each display, and to choose the equivalent expression from four possible choices. It was noted that all expressions were in degrees. They were instructed to respond 'quickly but still accurately' and were asked not to use paper and pen/pencil or a calculator and not to refer to any outside sources.

A single digit subtraction problem was presented as an example, followed by a block of problems. The order of trials in each block was random, and selected independently for each block. No feedback was provided. After the participant's response was recorded on each trial, the participant clicked to initiate the presentation of the next trial. Response times (from the presentation of the problem to the mouse click indicating the participant's choice response) were recorded on every trial. At the end of each block, participants produced a confidence rating (an estimated number of correct answers out of 40) and an open-ended description of how they solved the problems, with the specific instruction to describe anything they may have visualized and any rules, mnemonics or other strategies they may have used.

Additional self-reported measures were collected after completing the second block. First, participants rated on a five point scale (Not different at all, Slightly, Somewhat, Very, Extremely different) the extent to which the way they solved the second block of questions was different from the first block of questions. An open-ended response box was provided for participants to describe any changes in the way they solved the problems. Participants then estimated how recently (in years) they encountered trigonometry in school or in work (0 if current). They also estimated how many classes they had taken that involved trigonometry or required some use of trigonometric knowledge (with the instruction that this includes not only math classes, but also applications in the sciences and other areas). Participants were then asked to rate on a five point scale (Never, Rarely, Sometimes, Often, Always) how often, in solving each block of problems, they: (1) recalled an explicit rule or formula, (2) visualized the sine or cosine graphs as waves, (3) visualized sine and cosine as x and y coordinates of a circle, (4) visualized a right triangle with sine and cosine associated with sides of a triangle, (5) used a mnemonic (memorized acronym or phrase) to help remember facts about sine and cosine, or (6) used another representation or strategy. These ratings were first collected with participants instructed to consider only the first block, and were collected a second time after the participant was instructed to consider only the second block. Participants then rated how often they had used each representation in previous classes and other experiences with trigonometry, and they also rated how much they had been exposed to each representation.

Following this, 20 additional problems were used for problem-specific self-report assessment. These 20 problems included one example of every combination of function, sign of θ, and signed value of Δ, with randomly chosen values for θ and for the order of θ and Δ. Immediately after solving each problem, participants rated the extent to which they used each representation (1-6 above) on a three point scale (not at all, a little, a lot).

At the end of the study, we also asked students to solve three problems in front of the experimenter. The experimenter instructed students to talk through what they were thinking as they were thinking it, and this think-aloud protocol was recorded.[1] The three problems shown were: $\cos(-50+0)$, $\cos(20+180)$, and $\sin(90-70)$.

Due to a technical error, for the first five participants, one trial was not presented in their block of problem-specific reports, and one trial presented in their first main block was mis-specified, so that block was slightly unbalanced.

2.2.2 Results

General performance and background measures

Overall performance on the 80 problems in blocks 1 and 2 averaged only 52%, 95% bootstrapped[2] confidence interval (BCI) [46, 59]. Figure 2.4 shows performance broken down by problem type, collapsing across order (θ first or Δ first). (Order had no appreciable effect on accuracy in a logistic mixed model with a random intercept and slope for each student and for each problem type, $b = 0.04$, 95% CI $[-0.05, 0.12]$, $z = 0.88$, $p = .381$.) Accuracy on the trivial func$(\theta + 0)$ problems averaged 94% correct, 95% BCI [86, 97], and most (84%) participants answered all eight of these problems correctly (four with sin and four with cos).

[1] In gathering consent, we explained that recording was entirely optional. One student declined to be recorded.

[2] Bootstrapping was used due to its robust handling of issues such as the bounded, asymmetric and non-normal behavior of binomial proportions. We used 10,000 bootstrap runs to resample subjects, and confidence intervals have 95% point-wise coverage and are bias corrected and accelerated (BCa) (Efron, 1987)

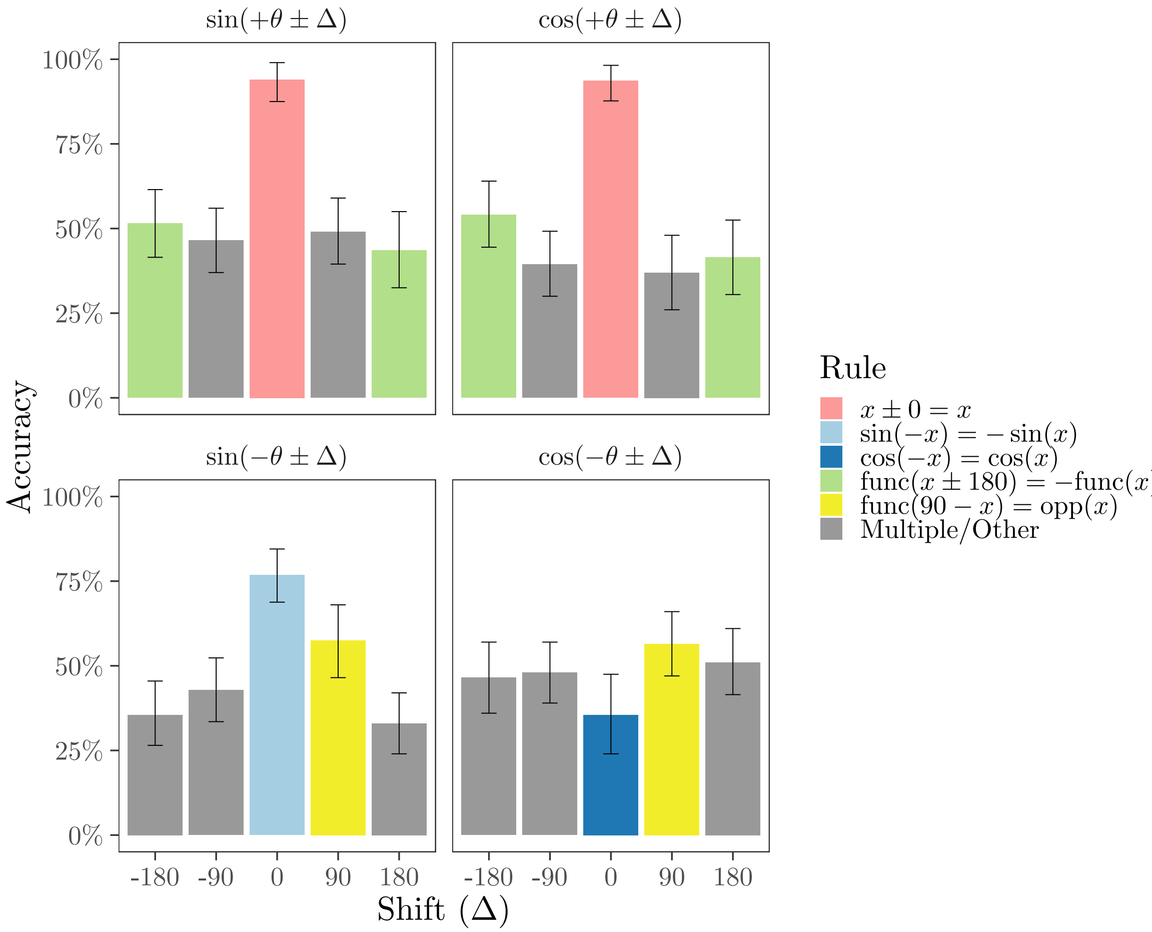


Figure 2.4: Mean accuracy (with 95% BCIs) split by problem type. The color indicates what trigonometric identity, possibly taken together with basic algebraic principles like $x \pm 0 = x$, can solve each problem type. Cases labeled Multiple/Other could be solved by a combination of these trigonometric and algebraic rules.

We next consider the eight types of problems that can be solved by a single application of one of the trigonometry rules shown in the top right of Figure 2.2 together with the basic principles of algebra like $x \pm 0 = x$. These problems are: $\cos(-\theta + 0)$, $\sin(-\theta + 0)$, $\cos(\theta + 180)$, $\cos(\theta - 180)$, $\sin(\theta + 180)$, $\sin(\theta - 180)$, $\sin(90 - \theta)$, and $\sin(90 - \theta)$). Figure 2.4 shows performance for each problem type. Mean accuracy on these problems was only 52%, 95% BCI [45, 60], with accuracy on $\cos(-\theta + 0)$ averaging only 36%, 95% BCI [24, 48]. Participants reported an average of 3.8 prior classes involving trigonometry, 95% BCI [3.2, 4.8], $SD = 2.7$, and an average of 2.5 years since last use of trigonometry, 95% BCI [2.0, 3.0], $SD = 1.7$. Thus, our findings replicate the striking finding of our preliminary study, indicating that the ability to use knowledge of very basic aspects of trigonometry that can be captured in a very small number of rules is quite poor, even among students with fairly extensive prior exposure to trigonometry at a highly selective private university.

Self-reported use of representation

Retrospective self-report ratings for blocks 1 and 2 were averaged (there were no reliable differences between blocks, -0.02, 95% CI [-0.11, 0.08], $F(5, 294) = 0.73$, $p = .605$). Figure 2.5 shows the distribution of ratings for each representation. There were significantly different levels of self-reported use between representations, Kruskal-Wallis $\chi^2(5) = 55.78$, $p < .001$. With a median rating of 'often' (4 on a 5 point scale), the unit circle had significantly more self-reported use than every other representation: the waves ($p < .001$), the right triangle ($p = .018$), rules/formulas ($p = .005$), mnemonics ($p < .001$), and others ($p < .001$) (pairwise Mann-Whitney tests corrected by Holm's procedure).

Exposure and prior use We next examined whether prior exposure to or prior use of the unit circle might explain the high degree of reliance on the unit circle in our study. The distribution of these ratings for each representation is shown in Figure 2.5. A Kruskal-Wallis one-way analysis of variance showed significant variation in self-reported exposure, Kruskal-Wallis $\chi^2(5) = 120.79$, $p < .001$. Students reported significantly less exposure to mnemonics ($p < .001$) and 'other' representations ($p < .001$) than the unit circle (pairwise Mann-Whitney tests corrected by Holm's procedure). However, there were no significant pairwise differences in reported exposure between the unit circle, rules, the right triangle, and waves, which all had a median rating of 'quite a bit' of exposure (4 on a 5 point scale).

The self-reported ratings of prior use tell a similar story. There was significant variation in self-reported prior use, Kruskal-Wallis $\chi^2(5) = 95.70$, $p < .001$; compared to the unit circle, students reported significantly less prior use of waves ($p = .003$), mnemonics ($p < .001$) and 'other' representations ($p < .001$). However, the median rating of prior use of the unit circle, rules, and the right triangle was 'often' (4 on a 5 point scale) and there were no significant pairwise differences in reported use of these representations. Thus, the exposure ratings showed that students were taught about other representations (including rules and formulae) that they could have used, and the prior use ratings showed that many had a large amount of experience using such representations.

Figure 2.5: Distribution of self-reported ratings of use of each representation / strategy during the current study (averaged across blocks 1 and 2), use during prior experiences, and exposure during prior experiences. The five-point scale for current and prior use was [Never, Rarely, Sometimes, Often, Always], and the five-point scale for exposure was [Not at all, A tiny bit, A little, Quite a bit, A lot].

Table 2.1: Parameters for a logistic mixed model using these self-reported measures of representation use and previous experience to predict whether each trial was answered correctly.

	Simple model		Model incl. UC experience	
Predictor	*b*	*p*	*b*	*p*
(Intercept)	-0.73 [-2.00, 0.55]	.266	0.67 [-1.05, 2.40]	.446
Rule / formula use	0.08 [-0.27, 0.43]	.652	0.04 [-0.29, 0.37]	.817
Unit circle use	-0.22 [-0.51, 0.07]	.139	-0.26† [-0.54, 0.02]	.066
Waves use	0.24* [0.05, 0.43]	.013	0.34** [0.11, 0.57]	.004
Right triangle use	0.22† [-0.03, 0.46]	.079	0.23† [0.00, 0.46]	.054
Mnemonic use	-0.22 [-0.61, 0.16]	.257	-0.41† [-0.84, 0.02]	.060
Other representation use	-0.11 [-0.38, 0.17]	.442	-0.11 [-0.37, 0.15]	.399
Number of relevant classes taken	0.11 [-0.02, 0.24]	.107	0.10 [-0.02, 0.23]	.109
Years since last trigonometry experience	-0.03 [-0.21, 0.15]	.725	-0.05 [-0.22, 0.12]	.554
Prior use of unit circle			0.16 [-0.30, 0.62]	.496
Exposure to unit circle			-0.46* [-0.91, -0.01]	.045

Unit circle use and overall accuracy

As a planned comparison, we examined whether reported unit circle use would be a better predictor of overall accuracy relative to other representations. We included each of the self-report representation ratings as well as classes and years since last exposure in a logistic mixed model to predict overall accuracy. Table 2.1 shows the fitted parameters of this model. Reported unit circle use significantly accounted for independent variance after taking into account all the other predictors, $b = 0.24$, 95% CI [0.05, 0.43], $z = 2.47$, $p = .013$; of the other self-report measures, only the right triangle predictor was marginally significant, $b = 0.22$, 95% CI $[-0.03, 0.46]$, $z = 1.76$, $p = .079$. After adding to our model self-reported prior use of and exposure to the unit circle, the unit circle remained the only significant predictor among the representation use ratings, $b = 0.34$, 95% CI [0.11, 0.57], $z = 2.92$, $p = .004$. This relationship is illustrated in Figure 2.6. Exposure to the unit circle was negatively related to accuracy, $b = -0.46$, 95% CI [-0.91, -0.01], $z = -2.00$, $p = .045$.

Further analysis suggests that many of those who reported using the right triangle might have used a representation with many of the properties of the unit circle. Of the eleven subjects who reported using the right triangle often or always six explicitly described their mental representation as a circle in their open-ended responses, and three other subjects described both a coordinate plane or axes and negative angles or quadrants. Thus, for most subjects who reported using the right triangle, their success may be explained in part by actual use of the unit circle or their use of a systematic representation of angles and trigonometric relationships on the (x,y) coordinate plane that is functionally similar to the unit circle.

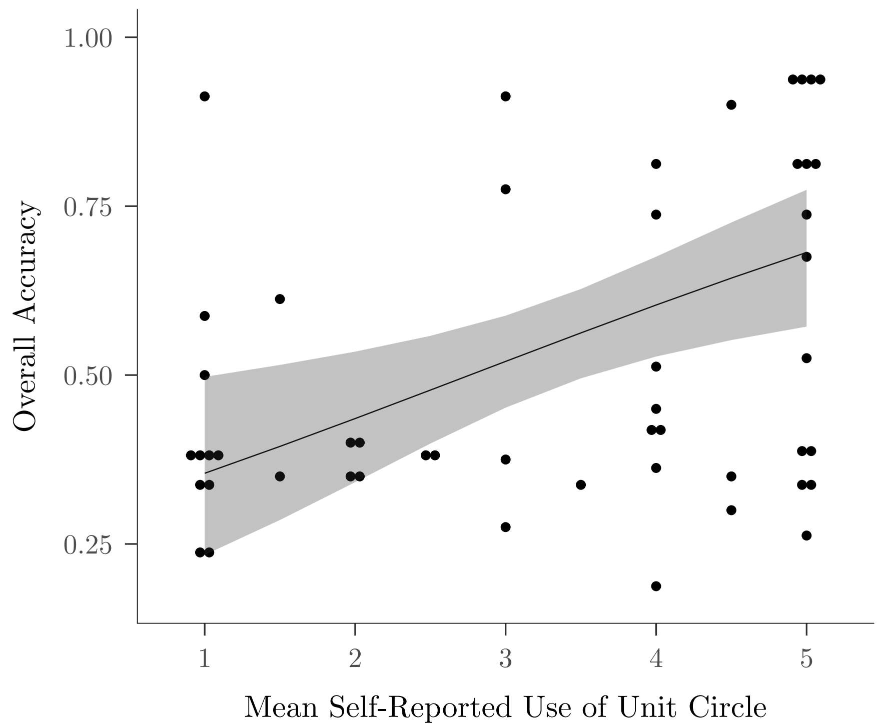


Figure 2.6: Mean accuracy by self-reported use of the unit circle (averaged over two blocks), with each point representing a student. The labels for each rating level were: 1='Never', 2='Rarely', 3='Sometimes', 4='Often', 5='Always'. The marginal predictions of the full model described in the text (which included as predictors: self-reported use of each representation, number of classes, years since last exposure, self-reported prior use of the unit circle and exposure to the unit circle) are shown along with its 95% confidence band.

Assessment of unit circle use on $\cos(-\theta + 0)$ **and** $\sin(-\theta + 0)$ **problems**

We next considered the relation between problem-specific ratings and performance for $\cos(-\theta + 0)$ and $\sin(-\theta + 0)$ problems. Figure 2.7 shows the distribution of responses to these problems, broken down by a student's problem-specific self-reported use of the unit circle. Looking only at these types of problems, we used a logistic mixed model to predict whether a student answered each problem correctly. The model included function (sine or cosine), self-reported use of the unit circle, and their interaction as fixed effects and also allowed these effects to vary between subjects. We included trials from the blocks 1 and 2 as well as the problem-specific report block, and we applied each student's circle rating to every similar problem in blocks 1 and 2. The interaction between problem type (specifically, function) and circle rating was significant, $b = -3.64$, 95% CI [−5.98, −1.31], $z = -3.06$, $p = .002$. We then examined the simple effect of self-reported use of the unit circle within each problem type. For the $\cos(-\theta + 0)$ problem type, the problem-specific unit circle rating predicts higher accuracy, $b = 4.62$, 95% CI [2.56, 6.67], $z = 4.40$, $p < .001$. In contrast, for $\sin(-\theta + 0)$ problems, we found no such relation, $b = 0.97$, 95% CI [−0.53, 2.48], $z = 1.27$, $p = .206$. The findings indicate that circle use was strongly associated with correct performance on $\cos(-\theta + 0)$, while performance on $sin(-\theta + 0)$ was generally accurate, independent of reliance on the unit circle. Among those reporting little or no reliance on the unit circle, the predominant error on $\cos(-\theta + 0)$ was $\cos(-\theta)$, accounting for 75% of errors on this problem type, 95% BCI [61, 85]. This error is consistent with a strategy of 'pulling out the minus sign,' which gives correct results for $sin(-\theta + 0)$, but not for $\cos(-\theta + 0)$.

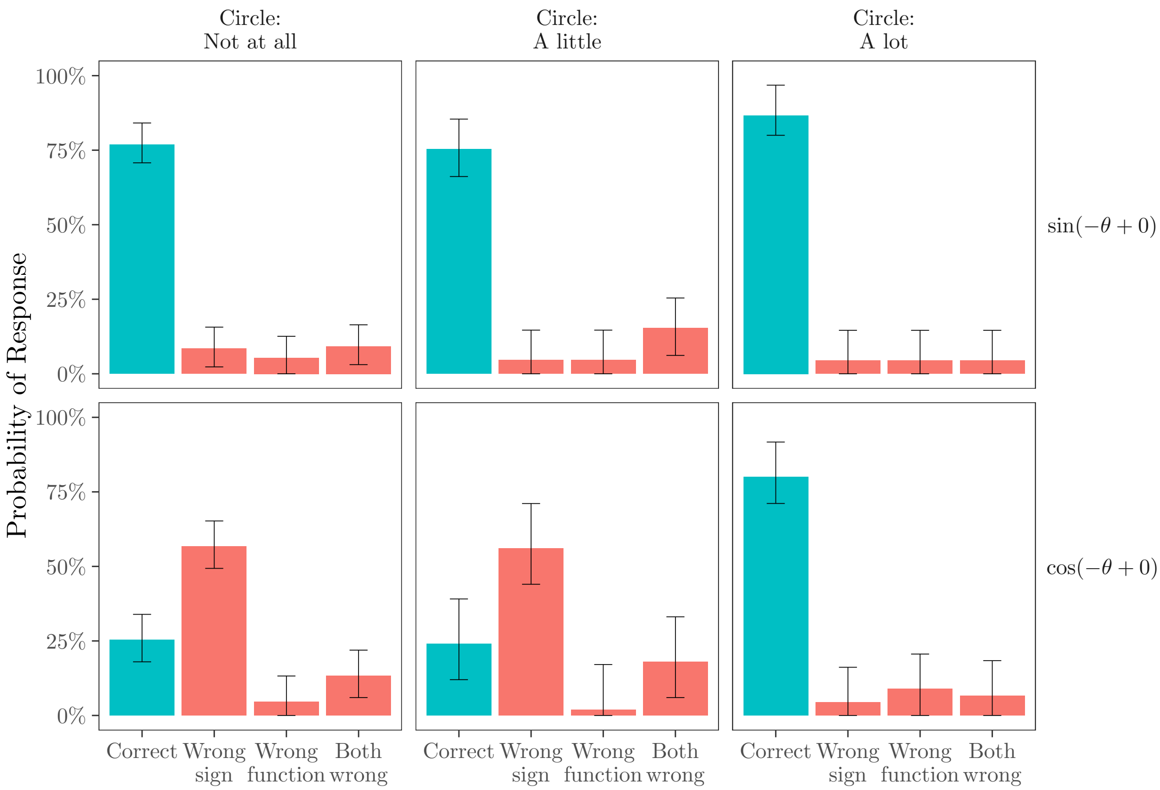


Figure 2.7: The distribution of responses to func($-\theta + 0$) with 95% simultaneous CIs, split by function and by level of unit circle use, based on problem-specific unit circle use rating.

2.3 Discussion

Study 1 finds highly consistent evidence that one particular visuospatial representation – the unit circle – plays a special role in helping students solve trigonometric identity problems, problems that could be solved in a number of different ways, involving both symbolic and spatial representations. The unit circle was the most commonly used representation in our task, and those students who reported using it tended to have higher overall accuracy. Use of the unit circle proved to be especially strongly associated with successful performance on $\cos(-\theta+0)$ problems. In spite of the simplicity of the symbolic expression for these problems and the centrality of the cosine function in trigonometry, performance was quite poor, and the predominant response was to choose $-\cos(\theta)$, rather than $\cos(\theta)$, as the answer. One explanation for this error is that participants may have a tendency to follow a simple heuristic strategy which could be described as 'pulling out the minus sign' – something that is consistent with symbolic rules in some situations (e.g. $(-A) = -(A)$), and happens to work for $\sin(-\theta + 0)$, but is not generally valid when dealing, as here, with a minus sign found in the argument to a function. If, on the other hand, one visualizes the cosine of a negative angle on the unit circle and compares this to the cosine of the corresponding positive angle, the identity of the corresponding values is apparent. Since no symbolic expressions are involved, there is no temptation of over-apply a 'pull out the minus sign' rule.

Our findings suggest the possibility of a causal relationship between circle use and effective trigonometry performance and motivate the future exploration of such a relationship through direct manipulation of knowledge and/or use of the unit circle in trigonometric reasoning. Accordingly, in Study 2, we devised two lessons, one grounding trigonometric relationships in the unit circle, and one promoting the use of rules without grounding them in any visuospatial conceptual structure. If, indeed, reliance on the unit circle facilitates solving trigonometric identities, then the grounded lesson should lead to more robust improvements than the formal lesson.

2.4 Study 2: Comparing a Formal Lesson with a Grounded Lesson

2.4.1 Method

Participants

We recruited 70 Stanford University undergraduate students to participate in a single hour-long session in exchange either for credit in an introductory psychology class or for pay ($12).

Procedure

The procedure was identical to that of Study 1, except that participants received either a grounded lesson or a formal lesson between blocks 1 and 2 of the experiment. The lessons were constructed to provide exposure to a parallel sequence of lesson elements, with each element presented on a single computer screen. One lesson presented each relationship in the context of the unit circle and the other presented it in the context of symbolic rules and manipulations of symbolic expressions (see Supplement for full lesson details). The set of lesson elements consisted of two subsets. The first subset dealt with the arguments to the sin and cos functions, including compound expressions, such as $(70 + 90)$. In the grounded lesson, this expression was presented as describing a sequence of angular rotations of a radial line shown on an accompanying diagram, in which one component corresponded to a 'special angle' (90) along with another 'arbitrary' angle, in this case 70. It was noted that rotations performed in either order produce the same result, so that expressions such as $(70 + 90)$ were equivalent to expressions such as $(90 + 70)$. In the formal lesson, this expression was presented as an instance of an expression involving an arbitrary angle that could be represented in a rule with a variable (x) along with a special angle, so that $(70 + 90)$ could be seen as an instance of the expression $(x+90)$. It was noted that principles of ordinary arithmetic apply to such expressions, so that a rule involving $(x + 90)$ it applies equally to expressions like $(70 + 90)$ and expressions like $(90 + 70)$.

The second subset of lesson elements covered eight of the 20 trigonometric identities included in our trigonometric identities test. The eight consisted of four pairs: $\sin(x + 0) = \sin(x)$ and $\cos(x + 0) = \cos(x)$; $\sin(-x + 0) = -\sin(x)$ and $\cos(-x + 0) = \cos(x)$; $\sin(x + 90) = \cos(x)$ and $\cos(x + 90) = -\sin(x)$; $\sin(x + 180) = -\sin(x)$ and $\cos(x + 180) = -\cos(x)$. From a formal, algebraic point of view, the set of rules together with principles of ordinary arithmetic were sufficient to solve all of the identity problems, although in some cases, more than one rule had to be applied to obtain the correct answer. For example, for the problem cos(-40-180), one must first apply the rule $\cos(-x) = \cos(x)$ to obtain $\cos(40 + 180)$, then one can apply the rule $\cos(x + 180) = -\cos(x)$ to obtain $-\cos(40)$.

Each of the identities was presented in the context of a specific problem, such as cos(50+0). For the formal lesson, a rule was introduced such as cos(x+0)=cos(x). The participant was then required to apply the rule to the given expression to derive the equivalent simplified expression, in this case cos(50), then choose this expression from the simplified-expression alternatives, in this case sin(50), -sin(50), cos(50), -cos(50). Participants could not move to the next screen until the correct expression was selected. For the grounded lesson, the principle captured by the rule was introduced by depicting the given expression as a radial line on a unit circle and the corresponding projection of that line on the horizontal (for cosine) or vertical (for sin) axis. In both cases the general principle was stated, and participants were required to apply the principle to the given problem to select the correct answer before moving on to the next lesson element.

At the end of each subset of lesson elements, participants rated their prior familiarity, understanding, and expected ability to apply the material to problems like those encountered in the first block. Participants in both groups then saw a final screen, stating that the rules (formal lesson) or relationships (grounded lesson) they had seen would apply directly to some of the problems they saw in the first part of the experiment, and would see again in the next part. They were also told that the rules or relationships would be helpful with other problems as well, but might need to be adapted or extended to address all of the problems. Formal lesson participants were told that "other rules from arithmetic and simple algebra might help you deal with some of the cases", while grounded lesson participants were told, "if you can visualize the given angle as a point on the unit circle and then visualize its x or y co-ordinate, and if you can do the same with the alternative answers, this will help you solve the problem."

With the addition of the lessons, we did not include a think-aloud protocol at the end of study, in order to keep the whole study within time constraints.

2.4.2 Results

General performance and background measures

Overall performance on the 80 problems in blocks 1 and 2 averaged 57%, 95% BCI [51, 62]. Accuracy on the trivial func($\theta + 0$) problems averaged 89% correct, 95% BCI [84, 92], and the majority (66%) of participants answered all eight of these problems correctly. Participants reported an average of 4.6 prior classes, 95% BCI [3.9, 6.1], $SD = 4.2$, and an average of 2.4 years since last use of trigonometry, 95% BCI [1.9, 2.9], $SD = 2.0$.

Comparability of lessons

Halfway through the lesson and again at the end of the lesson, each student rated how familiar the lesson material was, how much they understood, and how well they may be able to apply the material. Using Mann-Whitney tests corrected by Holm's procedure, there were no significant differences in the rated familiarity, understanding, or ability to apply between lessons. In the first section of both lessons, the median familiarity was 4.5 (between "Mostly" and "Completely"), the median understanding was 5 ("Completely"), and the median ability to apply was 4 ("Mostly"). In the second section of both lessons, the median familiarity was 3 ("Partially"), the median understanding was 4 ("Mostly"), and the median ability to apply was 4 ("Mostly").

We can also compare lessons by looking at how students rated their own confidence after solving the first block (before the lesson) and after solving the second block (after the lesson). In a one-way analysis of variance, we found that lesson condition does influence change in confidence rating, $F(2, 117) = 6.70$, $p = .002$. After block 1, students estimated how many problems they solved correctly (out of 40 problems), and the average across lesson condition was 15.23, 95% BCI [13.48, 17.09], $SD = 10.15$. Students with no lesson increased their estimated accuracy from block 1 to block 2 by 3.12, 95% CI [0.76, 5.48]. With contrasts comparing Study 1 (no lesson) to both lessons, students in both lessons showed significantly higher confidence than students with no lesson, $t(117) = 3.61$, $p < .001$. Students in the formal lesson increased their estimated problems solved correctly by 8.14, 95% CI [5.32, 10.97], and students in the grounded lesson increased their estimate by 9.37, 95% CI [6.55, 12.19]. With a contrast comparing the grounded lesson to the formal lesson, there was no significant difference in their effect on the change in confidence, $t(117) = 0.61$, $p = .543$.

Students rated the extent to which they changed their strategy or use of representations from block 1 to block 2 on a five point scale ("Not different at all", "Slightly", "Somewhat", "Very", "Extremely different"). Students in the first study with no lesson reported a median "slight" change, $M = 2.04$, 95% CI [1.70, 2.38], while students in the formal lesson condition and in the grounded lesson both had a median self-report of changing "somewhat", (Formal: $M = 3.14$, 95% CI [2.74, 3.54]; Grounded: $M = 2.77$, 95% CI [2.37, 3.17]). In a one-way analysis of variance, condition had a significant effect on rated change in representation or strategy, $F(2, 117) = 9.36$, $p < .001$. With a contrast comparing the first study (no lesson) to both lessons, students in both lessons reported significantly more change in representations relative to students with no lesson, $t(117) = 4.13$, $p < .001$, but in a contrast comparing the grounded lesson to the formal lesson, there was no significant difference, $t(117) = -1.30$, $p = .198$.

Effect of lessons on accuracy

We used a logistic mixed model to predict whether each trial was answered correctly. We included a random intercept for each participant, as well as a random effect of block. We included data from both Studies 1 and 2, coding condition with two contrasts: general lesson effect (both lessons vs. no lesson), and lesson type (grounded lesson vs. formal lesson). We tested the interaction of condition, block, and transfer (vs. taught).

Table 2.2 presents this simple model's fitted parameters. The interaction between block and general lesson was significant, $b = 0.06$, 95% CI [0.01, 0.12], $z = 2.28$, $p = .023$. The no lesson group did show some improvement from block 1 to block 2, $b = 0.39$, 95% CI [0.14, 0.64], $z = 3.07$, $p = .002$. This unsupervised practice effect has several potential explanations: perhaps students were gradually sharpening recollection of trigonometry, or perhaps students were adapting their knowledge to suit the structure of the problems over time. However, both groups with lessons showed stronger improvement, $b = 0.76$, 95% CI [0.11, 1.41], $z = 2.28$, $p = .023$. The formal lesson group improved significantly in accuracy from block 1 to block 2, $b = 0.39$, 95% CI [0.14, 0.64], $z = 3.07$, $p = .002$. The grounded lesson group showed the strongest improvement from block 1 to block 2, $b = 0.39$, 95% CI [0.14, 0.64], $z = 3.07$, $p = .002$. The interaction between block and lesson type was significant, $b = 0.12$, 95% CI [0.01, 0.22], $z = 2.21$, $p = .027$. This interaction reflects that the grounded lesson produced significantly stronger improvement than the formal lesson, $b = 0.48$, 95% CI [0.05, 0.90], $z = 2.21$, $p = .027$. To summarize the pattern of results, while both lessons did help students (relative to no lesson), the grounded lesson helped students achieve higher accuracy than the formal lesson.

Table 2.2: Parameters for a logistic mixed model using block, lesson condition, and, for one model, transfer (vs. taught) to predict whether each trial was answered correctly.

	Simple model		Model including transfer	
Predictor	b	p	b	p
(Intercept)	0.17 [-0.07, 0.41]	.172	0.24† [-0.01, 0.49]	.062
Block	0.32*** [0.24, 0.40]	$<$ .001	0.36*** [0.28, 0.45]	$<$ .001
Transfer (vs. taught)			-0.18*** [-0.24, -0.12]	$<$ .001
Lesson (vs. none)	0.11 [-0.05, 0.27]	.194	0.14 [-0.03, 0.30]	.105
Lesson type (grounded vs. formal)	0.01 [-0.31, 0.32]	.968	-0.03 [-0.35, 0.29]	.862
Block × Transfer			-0.11*** [-0.17, -0.05]	$<$ .001
Block × Lesson	0.06* [0.01, 0.12]	.023	0.08** [0.03, 0.14]	.003
Block × Lesson type	0.12* [0.01, 0.22]	.027	0.09 [-0.02, 0.19]	.125
Transfer × Lesson			-0.08*** [-0.12, -0.04]	$<$ .001
Transfer × Lesson type			0.09* [0.02, 0.17]	.017
Block × Transfer × Lesson			-0.06** [-0.09, -0.02]	.003
Block × Transfer × Lesson type			0.09* [0.02, 0.17]	.011

Table 2.3: Mean accuracy (with 95% CI), broken down by lesson condition, block, and transfer (vs. taught).

	Taught problems		Transfer problems	
Lesson condition	Block 1	Block 2	Block 1	Block 2
None	0.44 [0.37, 0.53]	0.50 [0.42, 0.59]	0.44 [0.37, 0.51]	0.50 [0.42, 0.58]
Rules only	0.52 [0.43, 0.60]	0.71 [0.62, 0.79]	0.48 [0.39, 0.57]	0.51 [0.41, 0.61]
Circle-based	0.47 [0.38, 0.56]	0.67 [0.57, 0.75]	0.43 [0.35, 0.53]	0.59 [0.50, 0.68]

The grounded lesson's benefit may have been driven in part by particularly greater improvement in accuracy on novel problems. Both the grounded lesson and the formal lesson included eight identities: $\sin(x+0) = \sin(x)$ and $\cos(x+0) = \cos(x)$; $\sin(-x+0) = -\sin(x)$ and $\cos(-x+0) = \cos(x)$; $\sin(x+90) = \cos(x)$ and $\cos(x+90) = -\sin(x)$; $\sin(x+180) = -\sin(x)$ and $\cos(x+180) = -\cos(x)$. The first two identities involving $x+0$ were intended to be trivial, and empirically most students achieved near ceiling performance before the lesson. The 12 remaining problem types were held out as transfer problems: $\text{func}(x-180)$, $\text{func}(-x \pm 180)$, $\text{func}(x-90)$, and $\text{func}(-x \pm 90)$, where func can be sin or cos.

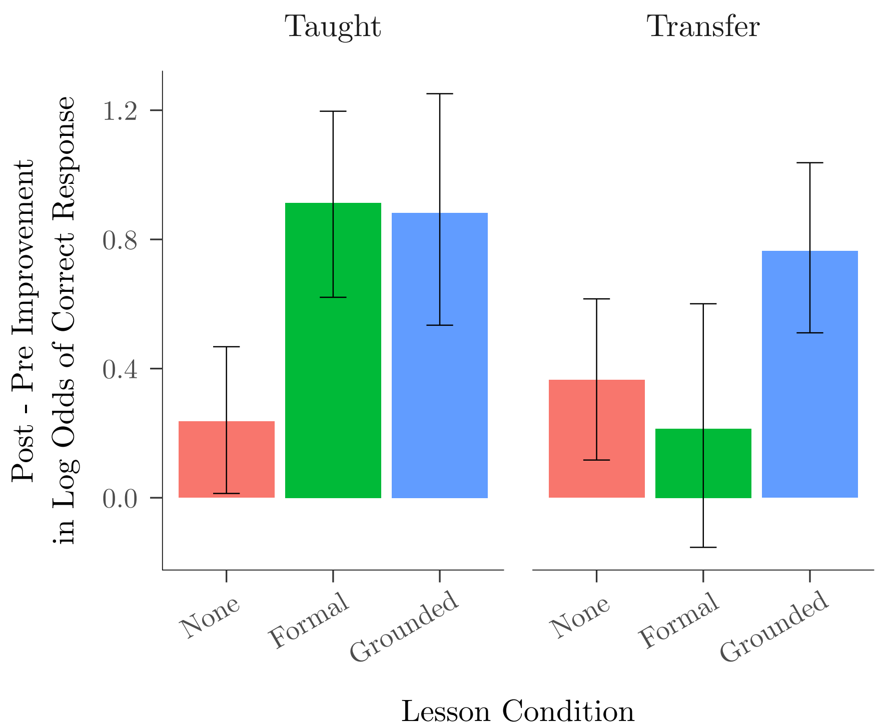


Figure 2.8: Mean improvement in log odds of responding correctly (with 95% BCIs) on taught vs. transfer problems, split by lesson condition.

Table 2.3 shows the mean accuracy for each lesson condition, broken down by block and transfer (vs. taught). To explore the potential interaction of lesson condition with taught versus transfer problems, we used a logistic mixed model to predict whether each trial was answered correctly. We included a random intercept for each participant, as well as a random effect of block, transfer, and their interaction. We included data from both Studies 1 and 2, coding condition with two contrasts: general lesson effect (both lessons vs. no lesson), and lesson type (grounded lesson vs. formal lesson). We tested the interaction of condition, block, and transfer. Note that because we assigned some problem types to be taught in the lessons and held out other problem types for transfer, we could still consider the contrast between taught and transfer problems in the no lesson group, in order to understand the baseline difference between different problem types.

Table 2.2 presents the fitted parameters of this full model, which considers taught versus transfer problems. The three-way interaction between block, transfer, and general lesson was significant, $b = -0.06$, 95% CI [-0.09, -0.02], $z = -2.95$, $p = .003$. While the no lesson group did show some improvement from block 1 to block 2 on problems assigned to be taught in the lessons, $b = 0.38$, 95% CI [0.07, 0.70], $z = 2.36$, $p = .018$, both lessons showed stronger improvement on taught problems, $b = 1.67$, 95% CI [0.83, 2.51], $z = 3.89$, $p < .001$. The effectiveness of the formal lesson on taught problems, $b = 1.24$, 95% CI [0.83, 1.64], $z = 6.01$, $p < .001$, and the effectiveness of the grounded lesson on taught problems, $b = 1.20$, 95% CI [0.81, 1.60], $z = 5.94$, $p < .001$, were not significantly different, $b = -0.04$, 95% CI [-0.59, 0.51], $z = -0.13$, $p = .898$.

The three-way interaction between block, transfer, and lesson type was also significant, $b = 0.09$, 95% CI [0.02, 0.17], $z = 2.55$, $p = .011$. Again, the no lesson group showed some limited improvement from block 1 to block 2 on problems assigned to be held out for transfer in the lesson conditions, $b = 0.40$, 95% CI [0.11, 0.69], $z = 2.72$, $p = .007$. This is comparable to the improvement on taught problems for the no lesson group, and it is due to the same practice effect (as nothing in the break between blocks distinguished these kinds of problems). Interestingly, there was no general lesson advantage for transfer problems over no lesson, $b = 0.33$, 95% CI [-0.43, 1.10], $z = 0.86$, $p = .390$. Instead, the grounded lesson showed significantly greater improvement than the formal lesson on transfer problems, $b = 0.72$, 95% CI [0.22, 1.21], $z = 2.84$, $p = .004$. Students who saw the formal lesson failed to show any significant improvement on transfer problems, $b = 0.21$, 95% CI [-0.14, 0.56], $z = 1.19$, $p = .234$. This suggests that a strictly rule-based approach works for cases where a rule can be applied, but struggles for cases where either multiple rules or a new rule are needed. On the other hand, students who saw the grounded lesson showed quite strong improvement on transfer problems, $b = 0.93$, 95% CI [0.58, 1.28], $z = 5.16$, $p < .001$. To summarize the pattern of results, the formal lesson led only to improvements on problem types that were included in the lesson, whereas the grounded lesson caused improvements on not only taught problem types but also problem types that were held out of the lesson to demonstrate transfer. Thus, the grounded lesson appears to be supporting some kind of general understanding or facilitates more flexible reasoning.

Effect of lessons on representation use

Our understanding of the effect of the lessons depends on the nature of the representation change. There are several possible explanations for why the grounded lesson supported successful problem solving and generalization of learning to novel problems. Visuospatial reasoning can provide students with an effective standalone procedure for deriving answers to problems. Or, students can be using both rules and visuospatial reasoning as independent strategies. Or, visuospatial reasoning can support the discovery, use, and memory of rules or rule-like representations to solve problems. What role or roles does the unit circle play in the minds of successful students? This question may not be easy to answer, but relevant self-reported measures may inform an answer. As expected, students who saw the formal lesson reported an increase in rule use from block 1 to block 2 and decreased self-reported use of the unit circle, the waves, and the right triangle. Interestingly, students who saw the grounded lesson reported increased use of both the unit circle and rules or formulae. They also decreased their self-reported use of the waves. Note that an increase in self-reported use of the unit circle was not significantly correlated or significantly anti-correlated with an increase in self-reported use of rules ($r = -.13$, 95% CI $[-.45, .21]$, $t(33) = -0.77$, $p = .448$). Therefore while there is variability between subjects in co-occurrence of strategies, the data did not seem to arise solely from a group of circle users and from a separate group of rule users.

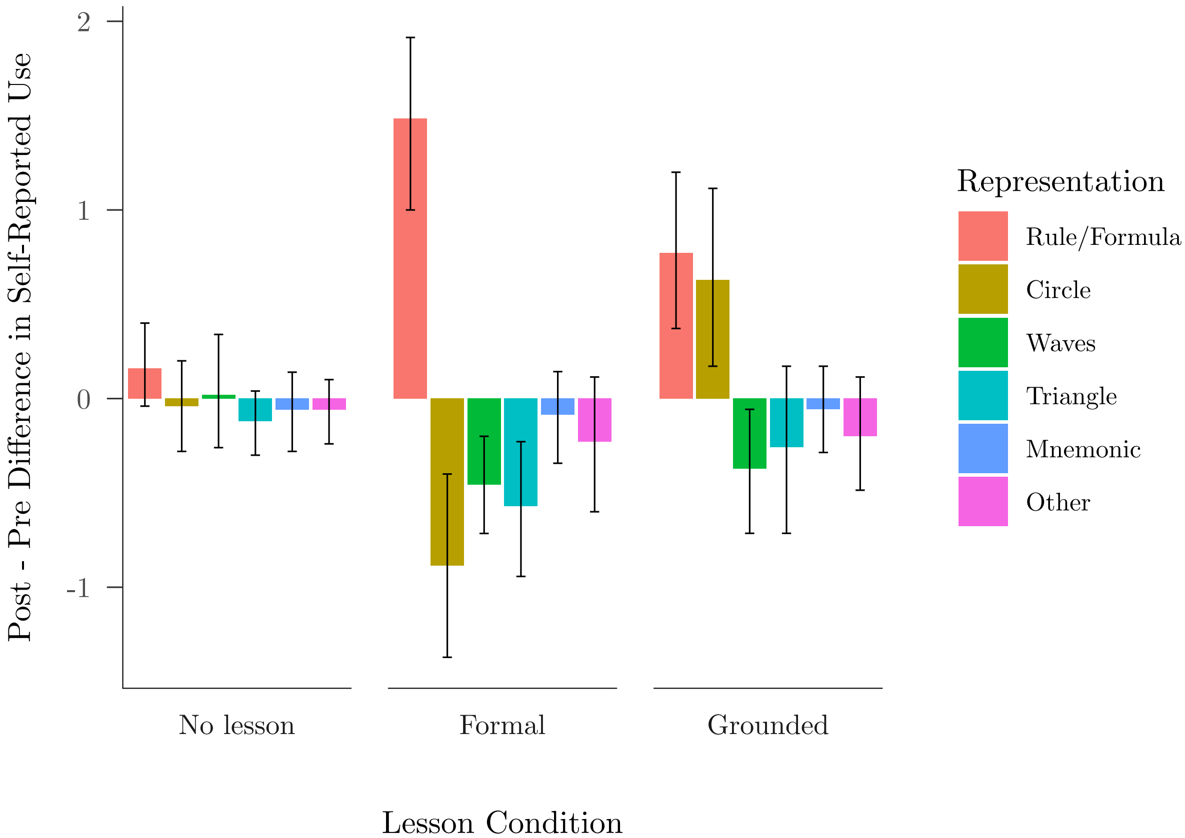


Figure 2.9: Mean change in self-reported use of each representation, by lesson condition. This change is measured as the difference between the block 1 ratings and block 2 ratings, each on a five point scale from "Never" to "Always."

2.5 Discussion

With students randomly assigned to the grounded lesson or the formal lesson, Study 2 offers experimental evidence consistent with the observation in Study 1 that successful students ground trigonometric relationships in the unit circle. In Study 2, we found that grounding trigonometric relationships in the unit circle can facilitate problem solving more effectively than relying on formal rules alone. Additional research is needed to fully elucidate the reasons why the unit circle seems well-suited for our task. Perhaps the unit circle is less arbitrary than other representations, so it grounds trigonometric concepts more effectively with conventions and relationships that may be already familiar to learners. Or perhaps the highly symmetric nature of the circle allows for easier computation. Or, in a very different role, the unit circle may be an introduction to mastering trigonometric concepts, and explicit reliance on a visuospatial representation can sometimes be short-circuited by reliance on a simple rule (such a pattern is frequently attested in other domains, e.g. Schwartz & Black (1996)). Note that the success of the grounded lesson should not necessarily be attributed to the unit circle alone, as students increased their self-reported use of both rules and the unit circle.

Importantly, the grounded lesson improved accuracy not only on problems taught in the lesson, but also problems held out for transfer. How did the grounded lesson help students achieve success on transfer problems? Our first hypothesis might be that the unit circle is a powerful and general standalone procedure. However, the grounded lesson did encourage greater use of both rules and the unit circle. If students are using both, what roles do rules and the circle play in solving trigonometric problems, and what is their relationship to each other as mental representations? A second hypothesis acknowledges the power of the unit circle, but includes rule-based reasoning as an additional, independent strategy used by students. Perhaps students use both on each problem as independent procedures racing to find an answer. Or perhaps students rely on rules to solve taught problems, and fall back on the unit circle to solve transfer problems. Our third hypothesis is that the unit circle provides a conceptual structure to make rules more meaningful, and students succeed with an understanding that involves intrinsically linked representations. In Chapter 3, I will describe a replication of the grounded lesson and its transfer benefits, and I will examine how students use rules and the unit circle to form a productive mathematical understanding of trigonometric relationships.

Chapter 3

Grounding Relationships in the Unit Circle Facilitates Transfer

While knowledge of mathematical relationships is necessary for successful problem solving, students must ultimately be able to apply their knowledge in flexible ways to novel problems. In this chapter I will argue that, indeed, grounding relationships in the unit circle facilitates much more than just successful problem solving; it enables students to understand the meaning of novel expressions and to utilize that understanding in novel problems.

To encourage productive reasoning, participants in both conditions of Study 2 saw a screen at the end of the lesson, stating that the rules (formal lesson) or relationships (grounded lesson) they had seen would apply directly to some of the problems they saw in the first part of the experiment, and would see again in the next part. They were also told that the rules or relationships would be helpful with other problems as well, but might need to be adapted or extended to address all of the problems. Formal lesson participants were told that “other rules from arithmetic and simple algebra might help you deal with some of the cases”, while grounded lesson participants were told, “if you can visualize the given angle as a point on the unit circle and then visualize its x or y co-ordinate, and if you can do the same with the alternative answers, this will help you solve the problem."

From a formal, algebraic point of view, the set of rules provided in the formal lesson together with principles of ordinary arithmetic were sufficient to solve all of the identity problems, although in some cases, more than one rule had to be applied to obtain the correct answer. For example, for the problem $\cos(-40 - 180)$, one must first apply the rule $\cos(-x) = \cos(x)$ to obtain $\cos(40 + 180)$, then one can apply the rule $\cos(x + 180) = -\cos(x)$ to obtain $-\cos(40)$. While these steps may seem to belong in an introductory algebra course, it may be difficult to recall the appropriate rule, hold it in working memory, and perform algebraic manipulations.

Indeed, students who saw our formal lesson in Study 2 did not show significant improvement on novel problems, whereas students who saw our grounded lesson did show positive transfer to problems not included in the lesson. Why did the grounded lesson succeed, where the formal lesson failed? There are a number of possible mechanisms through which the grounded lesson could facilitate transfer, and the exhortation at the end of the lesson allowed for multiple such mechanisms.

Grounding symbolic expressions in a meaningful domain may support flexible reasoning simply as a result of broader facility in that grounded domain. Glenberg et al. (2004) consider grounding to involve mapping symbols (or parts of a symbolic expression) to objects (real or imagined), and then meshing or combining these objects in a coherent simulation, guided by their constraints and affordances. Bransford et al. (1972) argue that constructing such a simulation is what underlies our interpretation of sentences and our memory of their semantic content. In mathematics, we can reference quantitative properties such as length and area of real or imagined objects, and by constructing a quantitative interpretation, we may be able to "make sense" of mathematical expressions (A. G. Thompson et al., 1994). In trigonometry, the unit circle can be used to find an approximate quantitative value for each symbolic expression.

Visuospatial reasoning can provide students with an effective standalone procedure for deriving answers to problems, but rules may have certain advantages. Having mastery of two different strategies may therefore help a student achieve both speed and accuracy. Indeed, students reported more use of rules after the formal lesson, while students who saw the grounded lesson reported more use of both rules and the unit circle. This suggests that rules do play an important role in problem solving, and that students were not simply using a unit circle-based quantitative interpretation.

While students may benefit from having rules as an additional independent strategy, the unit circle and rules may be more intimately related, through mutual support or interaction. The grounded lesson could change the way students discover, use, and remember rules or rule-like representations to solve problems. For instance, the symbolic rule $\sin(x + 180) = -\sin(x)$ may be verbally encoded as “when an angle is rotated halfway around the circle, the y-coordinate of its endpoint has the same magnitude but the opposite sign”. Note that this could facilitate generalization or transfer to -180 problems if they too are coded as “halfway around the circle”.

To investigate more fully how the grounded lesson facilitated transfer, we conducted a study that manipulated problem types used as taught or transfer, and that collected problem-specific ratings of representation use. The pattern of representation use across taught and transfer problems will allow us to evaluate some of the transfer mechanisms identified above.

3.1 Study 3: Identifying Transfer Strategies

3.1.1 Method

Participants

We recruited 64 Stanford University undergraduate students to participate in a single hour-long session in exchange either for credit in an introductory psychology class or for pay ($15 Amazon gift card).

Procedure

The procedure was very similar to the grounded condition of Study 2. Stanford undergraduates saw trigonometric expressions (e.g., $\sin(-70 + 180)$) and tried to identify which of four simpler expressions was equivalent to the given expression. There was one block of 40 problems, followed by a brief lesson grounding these relationships in the unit circle, followed by another block of 40 problems.

There were two key differences from previous studies. First, some of the problems were selected as counterbalanced taught/transfer problems. We kept $\theta + 0$ and $-\theta + 0$ problems as taught, and we included one sin/cos pair involving ± 90 and one sin/cos pair involving ± 180. Some students were randomly assigned to see func$(\theta - 180)$ instead of func$(\theta + 180)$, and some students were randomly assigned to see func$(90 - \theta)$ instead of func$(\theta + 90)$. This 2x2 design resulted in four different versions of the grounded lesson.

The second key difference was the addition of a block of problems with problem-specific self-reported strategies. These 20 problems included one example of every combination of function, sign of θ, and signed value of Δ, with randomly chosen values for θ and for the order of θ and Δ. Immediately after solving each problem, participants rated how confident they were that their answer to that problem was correct on a three point scale (not at all, somewhat, very). They then rated the extent to which they used each of several different representations on a three point scale (not at all, a little, a lot). This block was similar to the one in Study 1, except that half of the problems included the generic variable θ instead of some instantiated angle measure (e.g., $20°$). Within each of the four groups assigned to the different taught problem types, half the students saw the generic θ problems first, then the instantiated angle problems. The other half of students saw the instantiated angle problems first, then the generic θ problems.

At the end of the study, we also asked students to solve four problems in front of the experimenter. The experimenter instructed students to talk through what they were thinking as they were thinking it, and this think-aloud protocol was recorded.[1] The four problems shown were: $\cos(-50 + 0)$, $\cos(20 + 180)$, $\sin(90 - 70)$, and $\cos(\theta - 180)$.

3.1.2 Results

First we will examine whether this study in fact replicated the overall transfer effect of the grounded lesson in Study 2. After this replication, we will then consider how a student solves each different problem type, particularly when that problem type is held out of the lesson for transfer. Then, we ask what problem-specific representations students used in order to solve taught versus transfer problems. We also will explore think-aloud protocols of students solving a few problems, which will provide some limited qualitative evidence for the nature and use of grounded rules. Finally, we will examine the effects of the grounded lesson on $\cos(-\theta + 0)$, a problem with which many students struggled before the lesson.

General performance and background measures

Accuracy on the 40 problems in block 1 averaged 53%, 95% BCI [48, 59], $SD = 0.22$. Accuracy on the trivial func$(\theta + 0)$ problems averaged 91% correct (95% BCI [84, 95]), and most (78%) of participants answered all eight of these problems correctly. Participants reported an average of 5.5 prior classes, 95% BCI [4.6, 6.6], $SD = 4.0$, and an average of 2.3 years since last use of trigonometry, 95% BCI [1.8, 2.8], $SD = 2.0$.

[1]In gathering consent, we explained that recording was entirely optional. One student declined to be recorded, and another student asked to be recorded in audio but not video.

Overall transfer

We replicated Study 2's finding that the grounded lesson facilitates transfer of a student's understanding to problems not included in the lesson materials. Figure 3.1 compares performance on the problems we chose to counterbalance as taught or transfer problems, and it also shows the performance on those same problems in Studies 1 and 2, for reference. Focusing on Study 3 and looking only at its counterbalanced problems, we used a logistic mixed model to predict whether each trial was answered correctly. We tested the interaction of block and transfer (vs. taught), and we included a random intercept for each participant, as well as a random effect of block, transfer, and their interaction. Table 3.1 shows the parameters of this simple model. The interaction between block and transfer was significant, $b = -0.32$, 95% CI [−0.57, −0.07], $z = -2.55$, $p = .011$. Specifically, within the first block, there was no significant difference in accuracy between taught and transfer problems, $b = 0.10$, 95% CI [−0.15, 0.35], $z = 0.79$, $p = .430$, but within the second block, students solved more taught problems correctly than transfer problems, $b = -0.36$, 95% CI [−0.65, −0.07], $z = -2.40$, $p = .016$. However, students did still show significant improvement on transfer problems from block 1 to block 2, $b = 0.47$, 95% CI [0.26, 0.67], $z = 4.42$, $p < .001$.

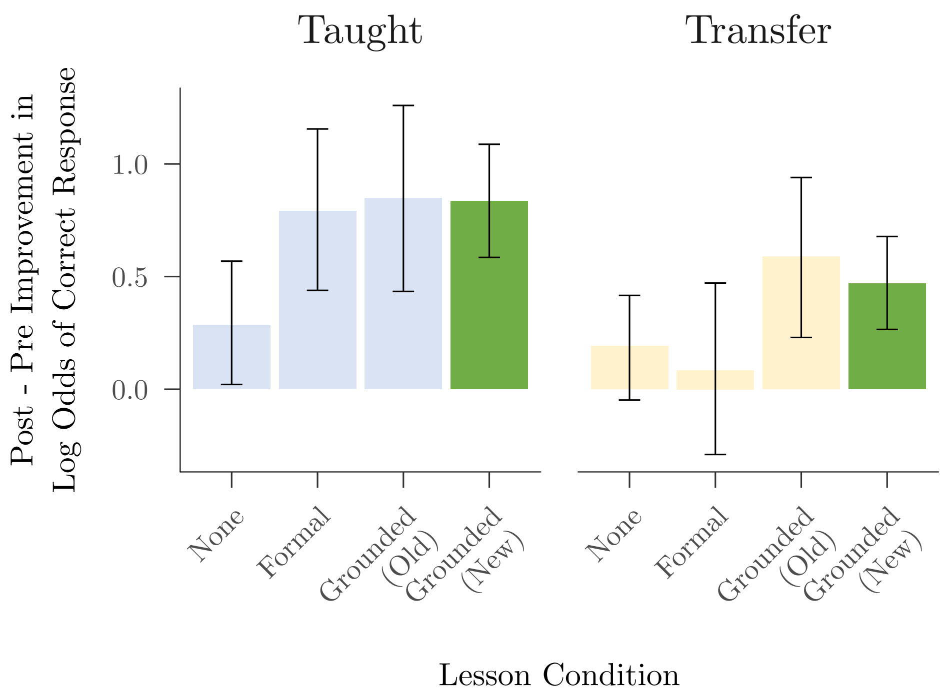


Figure 3.1: Mean improvement in log odds of responding correctly (with 95% BCIs) on taught vs. transfer problems, split by lesson condition. The data is restricted to the problems counterbalanced in this study, so that taught and transfer performance can be compared fairly. These problems were: func($\theta+180$), func($\theta-180$), func($\theta+90$), and func($-\theta+90$). In order to compare this with the data from previous studies, we restricted the old data to these problem types as well.

Transfer by problem type

An important question is whether transfer benefits of the grounded lesson are specific to only a few problems or are more general. While there are differences between problems, I will argue that the grounded lesson supports a student's ability to solve a broad range of problems.

To investigate the potential moderation of transfer by problem type, we used the same logistic mixed model as before, but added problem type as a fixed effect, along with its interactions with block and transfer (vs. taught). Again, we looked at only the problems that were counterbalanced in Study 3: func($\theta + 180$), func($\theta - 180$), func($\theta + 90$), and func($90 - \theta$). We analyzed problem type

Table 3.1: Parameters for a logistic mixed model using block, transfer, and (for one model) problem type to predict whether each trial was answered correctly. This model considers only problems which were counterbalanced as taught in the lesson or held out for transfer: func($\theta + 90$) or func($90 - \theta$), and func($\theta + 180$) or func($\theta - 180$).

	Simple model		Model with problem type	
Predictor	*b*	*p*	*b*	*p*
(Intercept)	0.62*** [0.29, 0.95]	$< .001$	0.64*** [0.30, 0.97]	$< .001$
Block	0.69*** [0.52, 0.87]	$< .001$	0.70*** [0.52, 0.87]	$< .001$
Transfer (vs. taught)	-0.13 [-0.33, 0.08]	.227	-0.09 [-0.28, 0.09]	.324
Shift (90 or ±180)			0.10 [-0.11, 0.31]	.347
90 type ($\theta + 90$ or $90 - \theta$)			0.73*** [0.50, 0.97]	$< .001$
180 type ($\theta + 180$ or $\theta - 180$)			-0.21† [-0.44, 0.02]	.080
Block × Transfer	-0.32* [-0.57, -0.07]	.011	-0.28* [-0.53, -0.04]	.025
Block × Shift			0.43** [0.13, 0.73]	.005
Block × 90 type			-0.12 [-0.44, 0.20]	.464
Block × 180 type			0.19 [-0.12, 0.51]	.227
Transfer × Shift			0.32* [0.02, 0.62]	.037
Transfer × 90 type			-0.15 [-0.56, 0.25]	.455
Transfer × 180 type			-0.25 [-0.65, 0.16]	.232
Block × Transfer × Shift			0.28 [-0.15, 0.70]	.202
Block × Transfer × 90 type			-0.02 [-0.47, 0.44]	.943
Block × Transfer × 180 type			-0.29 [-0.74, 0.16]	.213

with three orthogonal contrasts: the shift effect of ±180 vs 90, the difference between +180 and -180, and the difference between a positive and negative angle θ with a shift of 90.

Table 3.1 shows the parameters of this full model. None of the three-way interactions between block, transfer, and a problem type contrast were significant. There was a significant interaction between block and the shift effect (±180 vs 90), $b = 0.43$, 95% CI [0.13, 0.73], $z = 2.84$, $p = .005$, as well as a significant interaction between transfer and the shift effect, $b = 0.32$, 95% CI [0.02, 0.62], $z = 2.09$, $p = .037$. In block 1, there was no significant shift effect, $b = -0.21$, 95% CI [−0.49, 0.08], $z = -1.41$, $p = .160$, but after the lesson, in block 2, students solved problems with a shift of ±180 correctly more often than problems with a shift of 90, $b = 0.41$, 95% CI [0.10, 0.72], $z = 2.56$, $p = .010$. There was also no significant shift effect within taught problems, $b = -0.12$, 95% CI [−0.43, 0.19], $z = -0.79$, $p = .432$, but within transfer problems, students solved those with a shift of ±180 correctly more often than those with a shift of 90, $b = 0.33$, 95% CI [0.04, 0.62], $z = 2.21$, $p = .027$. Because the lesson occurred between blocks 1 and 2, and the lesson is what determines whether a problem is taught or held out for transfer, we would perhaps expect the three-way interaction between block, transfer, and shift effect to be consistent with transfer and its interactions only affecting block 2, and it was consistent although not significant, $b = 0.28$, 95% CI [−0.15, 0.70], $z = 1.28$, $p = .202$.

To understand why transfer was stronger for some problems over other problems, we must consider specific characteristics of different problems.

Transfer among 180 problems. Problems with a shift of ±180 seem well-suited for grounding in the unit circle. Figure 3.2, which includes results from Studies 1, 2, and 3, shows the change in accuracy on problems with a shift of ±180, from block 1 to block 2. While the formal lesson only facilitates improvement on its taught 180 problem, both variants of the grounded lesson successfully facilitate transfer across all problems with a shift of ±180. The equivalence of +180 degrees and -180 degrees is intrinsic in the definition of a circle having 360 degrees. This leads to students referring to 180 degrees as "a halfway turn" or "the opposite side of the unit circle," and as students learn a relationship involving +180 degrees or -180 degrees, their understanding allows them to naturally generalize that relationship to ±180 degrees.

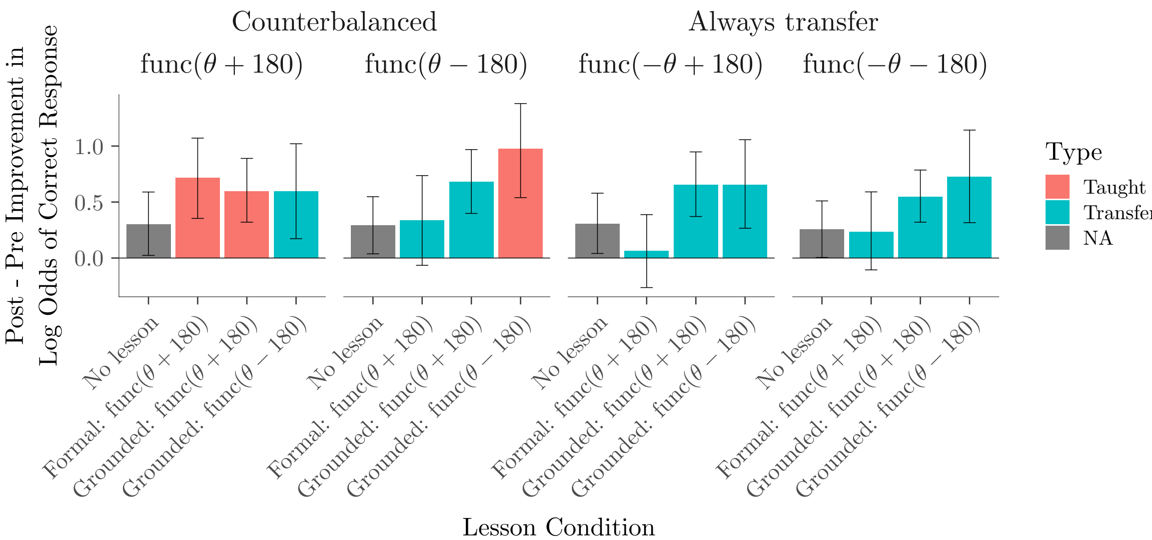


Figure 3.2: Mean improvement in log odds of responding correctly on 180 problems from block 1 to block 2 (with 95% BCIs), split by problem type and lesson condition. The grounded lesson condition was split based on the type of 180 problem presented in the lesson, with the grounded lesson of Study 2 grouped together with students in Study 3 who were taught func($\theta + 180$).

Transfer among 90 problems. Figure 3.3 displays the change in accuracy on problems with a shift of ±90, from block 1 to block 2. Again, the formal lesson only facilitates improvement on its taught 90 problem. Unlike problems with a shift of ±180, the 90 problems that were counterbalanced showed little or no transfer. The problems that were always held out of the lesson did, however, seem to show moderate transfer. It is possible that the problems we chose to counterbalance had specific characteristics that inhibited transfer.

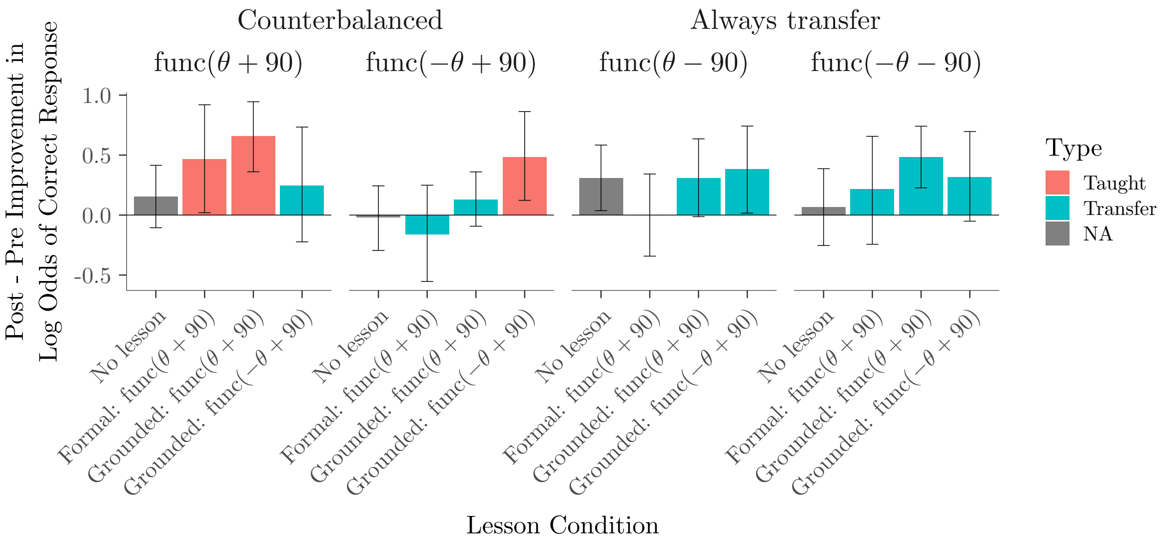


Figure 3.3: Mean improvement in log odds of responding correctly on 90 problems from block 1 to block 2 (with 95% BCIs), split by problem type and lesson condition. The grounded lesson condition was split based on the type of 90 problem presented in the lesson, with the grounded lesson of Study 2 grouped together with students in Study 3 who were taught func($\theta + 90$).

First, I will consider why the grounded lesson seemingly failed to facilitate transfer func($-\theta + 90$). These problems are unique due to the knowledge that students already possess prior to the lesson. Figure 3.4 shows the block 1 performance for each problem type, across Studies 1, 2 and 3. Unsurprisingly, students answered the trivial func($\theta + 0$) problems correctly the most often. The problem type with the next highest block 1 performance was func($-\theta + 90$).[2]

[2]The function in func($-\theta + 0$) has a particularly large effect on accuracy. If we were to split this problem by function, sin($-\theta + 0$) would have been higher ranking with mean block 1 performance of 76%, 95% BCI [70, 80], while on cos($-\theta + 0$) students had an average block 1 performance of 39%, 95% BCI [32, 45].

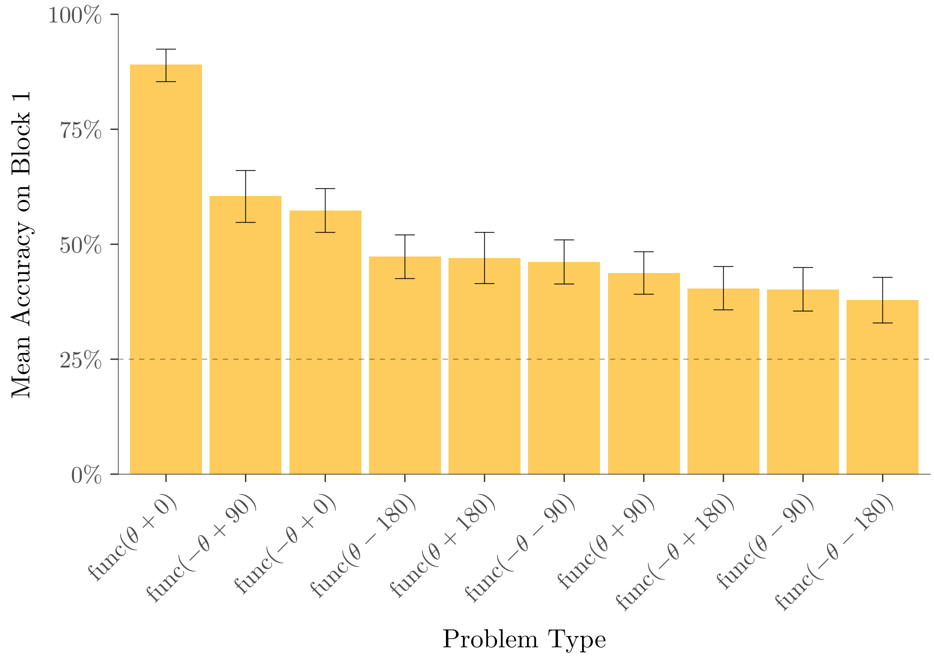


Figure 3.4: Mean accuracy on block 1 (with 95% BCIs) across all students from Studies 1, 2, and 3, broken down by problem type (collapsed across function).

In addition to having general strategies, some students retrieved specific strategies for solving func($-\theta$+90). The explicit rule that students most commonly reported in the open-ended description of strategies after block 1 was func($90 - \theta$) $=$ opp(θ) (or an instance thereof). Other students also reported using a triangle-based schema, consisting of complementary angles and knowledge of trigonometric functions as ratios of sides of a triangle. When the grounded lesson included func($\theta + 90$) problems and not func($-\theta + 90$), students may have adopted a more general approach based on the unit circle, and discarded or ignored their old strategies, which may not have seemed useful or relevant during the lesson. This scenario would result in roughly equal performance in block 2 on func($-\theta + 90$) and other transfer problems with a shift of 90. Because of the higher block 1 performance, though, the func($-\theta + 90$) problems would have little change in accuracy and therefore appear to show little or no transfer. The actual pattern of results, shown in Table 3.2,

Table 3.2: Mean accuracy (with 95% CI), broken down by block and problem type, for participants who saw func($\theta + 90$) in the lesson.

Problem type	Block 1	Block 2
func($\theta - 90$)	0.37 [0.28, 0.45]	0.47 [0.37, 0.56]
func($-\theta + 90$)	0.60 [0.50, 0.68]	0.63 [0.54, 0.71]
func($-\theta - 90$)	0.41 [0.33, 0.49]	0.57 [0.49, 0.64]

indeed involves greater improvement for func($\theta - 90$) and func($-\theta - 90$) than func($-\theta + 90$). However, this improvement was not quite enough to remove any difference in block 2. In an exploratory mixed logistic model of success on these problems on block 2, students were significantly more accurate on func($-\theta + 90$) than on func($\theta - 90$), $b = 1.02$, 95% CI [0.60, 1.44], $z = 4.75$, $p < .001$, or on func($-\theta - 90$), $b = 0.61$, 95% CI [0.20, 1.02], $z = 2.89$, $p = .004$.

For this explanation to be sufficient, we would also need to consider behavior when the grounded lesson instead included func($-\theta + 90$) as taught problems. At first, we might question why there was such strong improvement as taught problems (from a starting point already high in accuracy), when students could have "discarded or ignored their old strategies." During the lesson, however, in the course of learning to ground trigonometric expressions in the unit circle, students (assigned to this condition) saw the func($-\theta + 90$) problems, where their old strategies were relevant and potentially useful. Students could have integrated and grounded their old strategies in the unit circle. For instance, someone who used a right triangle to solve 90 degree problems may see the right triangle embedded in the unit circle representation, and these could provide mutual support in a virtuous cycle. They also received feedback, and so may have gained confidence in valid old strategies for these particular problems. These possible responses to the lesson are consistent with the improvement on taught func($-\theta + 90$) problems that students in this condition exhibited (shown in Figure 3.3).

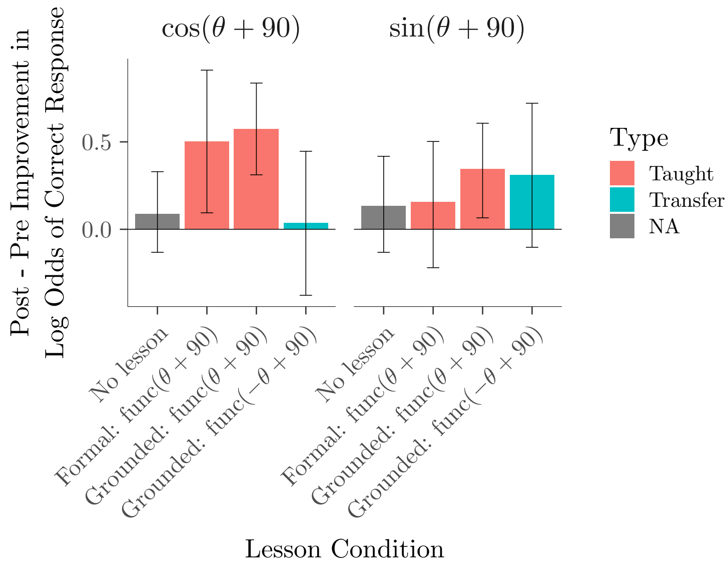


Figure 3.5: Mean improvement in log odds of responding correctly on func($\theta + 90$) problems from block 1 to block 2 (with 95% BCIs), split by function and lesson condition. The grounded lesson condition was split based on the type of 90 problem presented in the lesson, with the grounded lesson of Study 2 grouped together with students in Study 3 who were taught func($\theta + 90$).

The other problem type for which the grounded lesson seemingly failed to facilitate transfer was func($\theta + 90$). While I have previously been considering problem type by collapsing across function (cos and sin), performance on this problem type seems to be sensitive to function. Figure 3.5 shows improvement in accuracy by condition for this problem type, split by function. The grounded lesson appears to facilitate transfer on sin($\theta + 90$) at least to the same approximate magnitude as performance when taught. On cos($\theta + 90$), however, students show relatively large improvement when this problem is taught, and little to no improvement when this problem is held out for transfer.

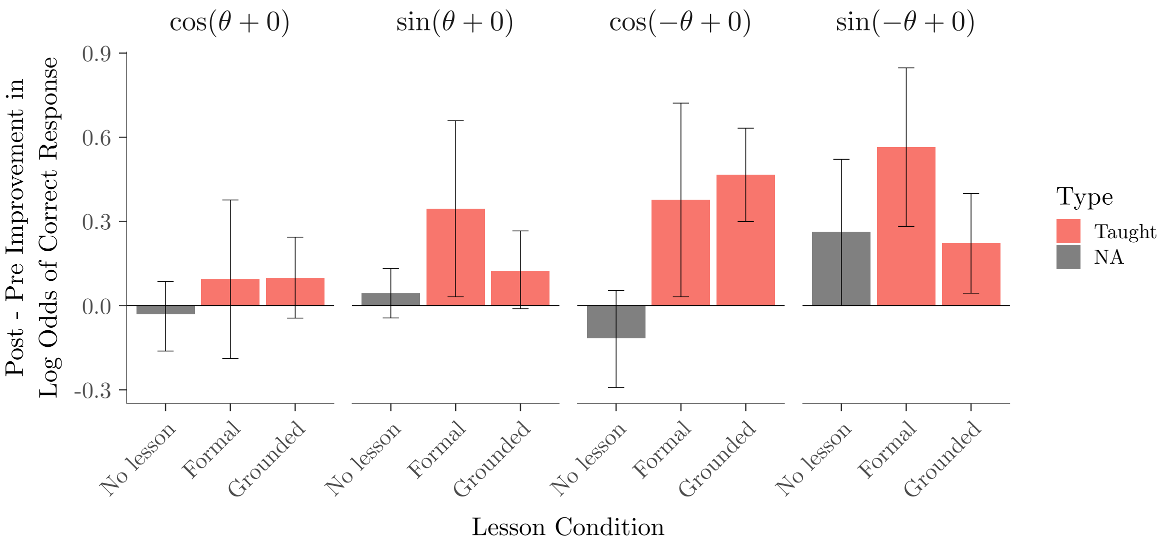


Figure 3.6: Mean improvement in log odds of responding correctly on 0 problems from block 1 to block 2 (with 95% BCIs), split by problem type and lesson condition. The grounded lesson condition includes the students assigned to the grounded lesson in Study 2 and all students in Study 3.

One possible explanation for this disparity rests on the idea that the brief grounded lesson is not a wholesale replacement of strategies employed by students. Just as some students may have retained valid old strategies for func$(-\theta + 90)$, some students may have also retained invalid old strategies. One common invalid strategy that students applied in block 1 was the “pulling out the minus sign” heuristic, discussed in Chapter 2. This heuristic was most apparent in the $\cos(-\theta + 0)$ problem, in which the correct answer was $\cos(\theta)$ but many students responded $-\cos(\theta)$. Figure 3.6 includes this problem, as well as other problems with a shift of 0, conveying the complete pattern of results in conjunction with Figures 3.2 amd 3.3. While the grounded lesson does support learning to solve the $\cos(-\theta + 0)$ problem, there are two pieces of evidence which may temper our view of this success. First, the formal lesson causes nearly the same magnitude of improvement in learning to solve this problem. Secondly, students in the grounded lesson continued to respond $-\cos(\theta)$ at an alarming frequency of 40%, 95% BCI [32, 47], $SD = 0.38$.

If students struggled with the sign of $\cos(-\theta+0)$, those struggles could extend to other problems in which the sign of the argument to the trigonometric function (in the original probe expression or in an intermediate expression) conflicts with the sign of the correct answer. For instance, the problem $\cos(\theta+90)$ involves a sign conflict because the correct answer $-\sin(\theta)$ involves a negative sign not present in the original expression.

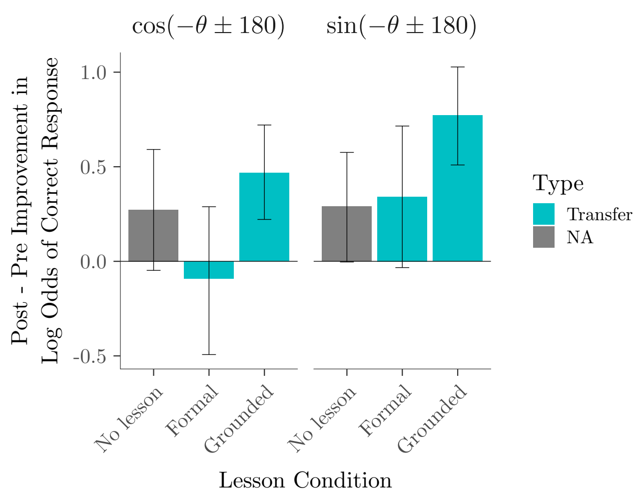


Figure 3.7: Mean improvement in log odds of responding correctly on func($-\theta\pm180$) problems from block 1 to block 2 (with 95% BCIs), split by function and lesson condition. The grounded lesson condition includes the students assigned to the grounded lesson in Study 2 and all students in Study 3.

Another problem type with a possible sign conflict is $\cos(-\theta \pm 180)$. The correct answer is $-\cos(\theta)$, but many students may be tempted to answer $\cos(\theta)$. They may be aware that ±180 results in changing the positive value to a negative value, and go on to reason (incorrectly) that the negative sign in front of θ reverses this result. Figure 3.7 shows performance on these problems as well as $\sin(-\theta \pm 180)$, broken down by lesson condition. While the grounded lesson does facilitate transfer for both cosine and sine, such facilitation appears to be weaker for cosine. (Note that this is also true in the formal lesson, albeit lower overall improvement in accuracy.) Or, in other words, the grounded lesson is facilitating transfer generally, although students are sensitive to characteristics of problems that may present difficulties.

Representations on taught and transfer problems

To investigate what mechanisms may underlie succesful transfer after the grounded lesson, we examined what representations students used on taught and transfer problems. Different types of problems may naturally be suited for different kinds of representations, but because we counterbalanced some of the problems included in the lesson, we can compare strategy differences between taught and transfer problems while controlling for problem type. Figure 3.8 shows this comparison, and it also provides a sense of the overall distribution of representations students reported using on each trial.

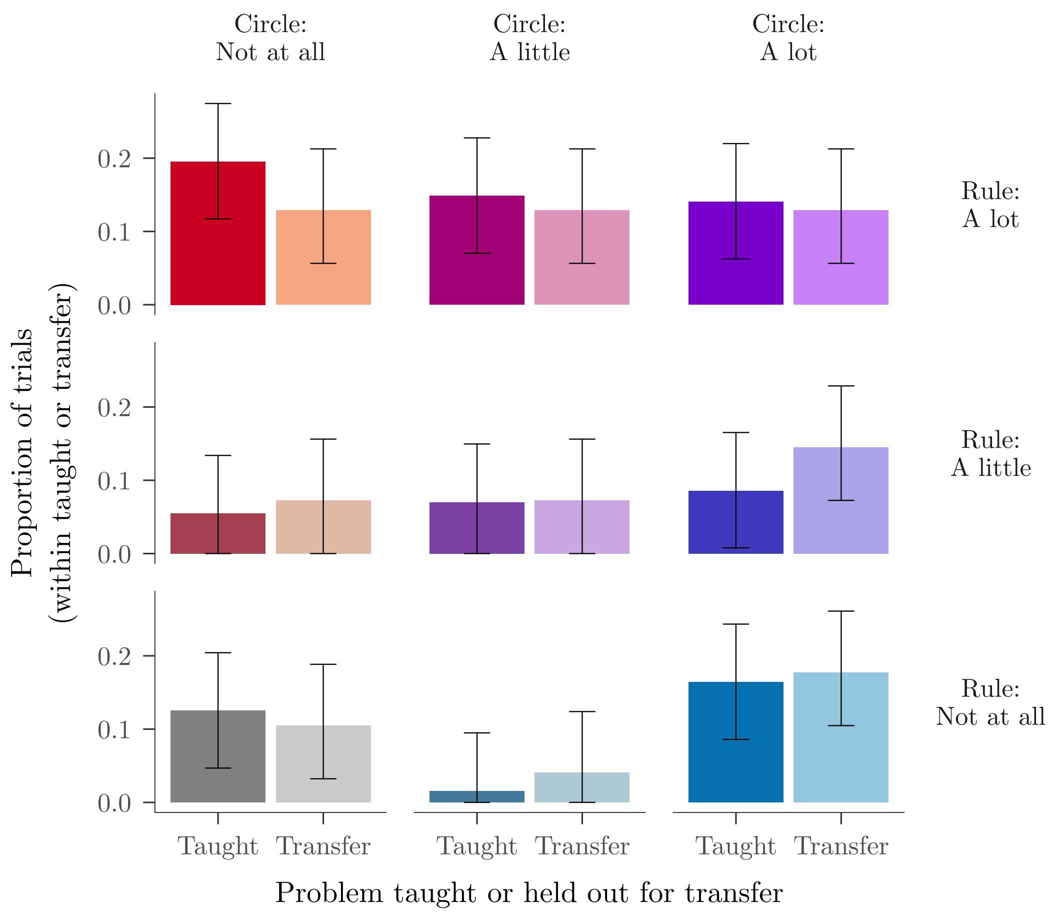


Figure 3.8: Porportion of trials for each combination of self-reported reliance on the unit circle and on a rule / formula, separately for problems taught in the lesson and for problems held out for transfer. This data is restricted to the problems counterbalanced in this study, and problems that contained a specific angle (as opposed to the generic variable θ), which matches the kind of problems seen in blocks 1 and 2.

Students rated their problem-specific use of each representation on a 3-point scale, so we used cumulative link mixed models to predict their ratings, restricted to only the counterbalanced problems. Each model tested the effects of transfer (vs. taught) and value type (specific angle vs generic theta) on problem-specific use of either the unit circle or a rule/formula. We included a random intercept for each participant, as well as a random effect of transfer, value type, and their interaction. We found that students reported relying significantly more on a rule or formula for taught problems, compared to transfer problems, $b = -0.27$, 95% CI [−0.51, −0.02], $z = -2.11$, $p = .035$. There was no significant effect of value type, $b = 0.06$, 95% CI [−0.22, 0.35], $z = 0.43$, $p = .664$, or interaction of value type with transfer, $b = 0.02$, 95% CI [−0.22, 0.25], $z = 0.13$, $p = .897$, for reliance on rules or formulae. For reliance on the unit circle, there was no significant effect of transfer, $b = 0.21$, 95% CI [−0.06, 0.47], $z = 1.52$, $p = .128$, or interaction of transfer with value type, $b = -0.16$, 95% CI [−0.40, 0.07], $z = -1.34$, $p = .180$. However, when students were shown a problem involving θ instead of some specific angle, they reported less reliance on the unit circle, $b = -0.48$, 95% CI [−0.74, −0.22], $z = -3.58$, $p < .001$. Note there was no significant correlation or anti-correlation between a student's mean self-reported reliance on the unit circle and mean reliance on a rule or formula, $r = -.15$, 95% CI [−.38, .10], $t(62) = -1.18$, $p = .242$.

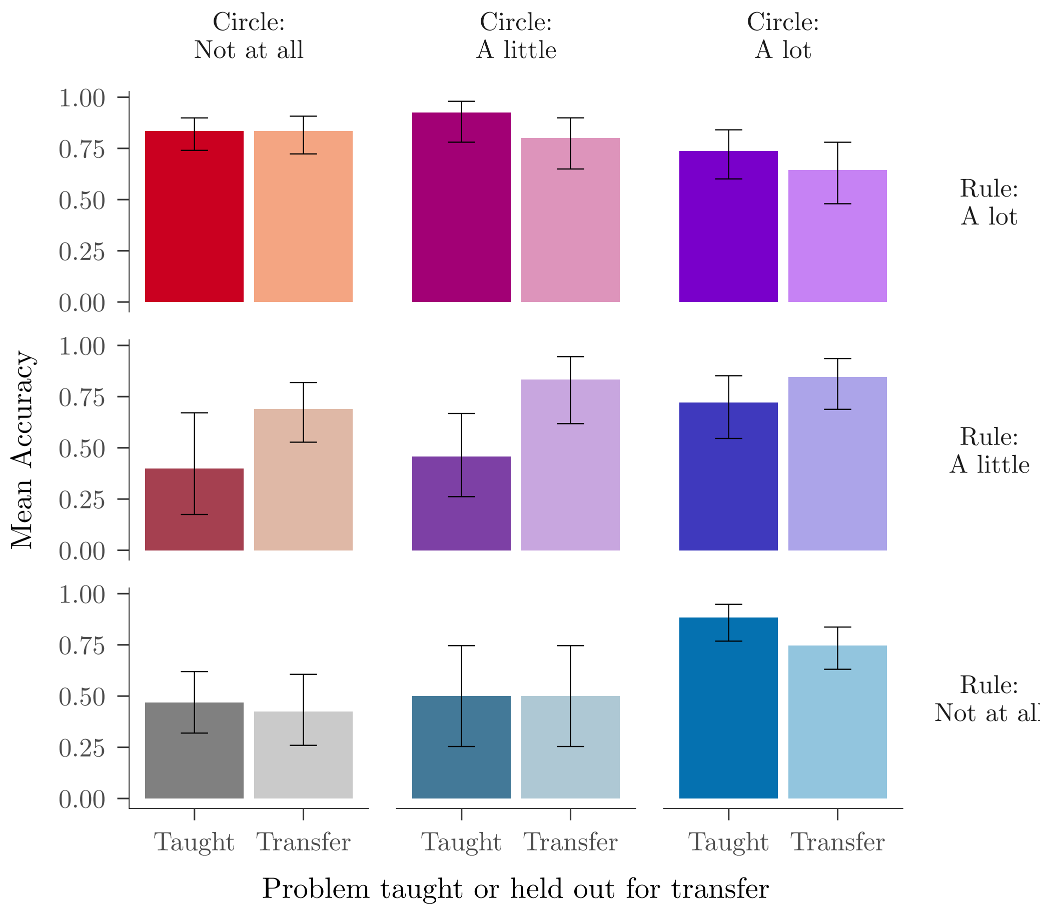


Figure 3.9: Accuracy on trials for each combination of self-reported reliance on the unit circle and on a rule / formula, split by whether the problem was taught in the lesson or held out as a novel transfer problem. This data is restricted to the problems counterbalanced in this study that included a shift of 180 or -180 degrees. We included trials from the self-report block as well as trials from block 2, whose strategy was coded as the strategy that student used on the same problem in the self-report block.

Beyond just frequency, we also want to understand the relationship between representations used and accuracy. Figure 3.9 shows the accuracy on trials for each combination of self-reported reliance on the unit circle and on a rule or formula, split by taught versus transfer. To analyze the effect of representation use and transfer on accuracy, we used a logistic mixed model with self-reported specific use of the unit circle, self-reported specific use of a rule or formula, and transfer (vs. taught)

Table 3.3: Parameters for a logistic mixed model using self-reported use of unit circle and rule or formula, transfer, and, for one model, confidence to predict whether each trial was answered correctly.

	Simple model		Model with confidence	
Predictor	*b*	*p*	*b*	*p*
(Intercept)	3.60** [1.33, 5.88]	.002	3.63** [1.10, 6.17]	.005
Self-reported specific use of unit circle	-0.92 [-2.91, 1.08]	.368	-1.08 [-3.16, 1.01]	.312
Self-reported specific use of rule / formula	2.37* [0.44, 4.30]	.016	1.50 [-0.61, 3.62]	.163
Transfer (vs. taught)	-0.14 [-0.62, 0.35]	.584	-0.19 [-0.90, 0.52]	.604
Self-reported confidence			3.32*** [1.53, 5.12]	$< .001$
Circle × Rule/Formula	-1.72* [-3.42, -0.01]	.048	-1.53 [-3.41, 0.36]	.112
Circle × Transfer	-0.14 [-0.72, 0.44]	.641	0.50 [-0.33, 1.33]	.236
Rule/Formula × Transfer	-0.19 [-0.76, 0.38]	.512	0.06 [-0.72, 0.84]	.874
Circle × Confidence			-0.43 [-2.47, 1.61]	.679
Rule/Formula × Confidence			1.14 [-0.45, 2.74]	.159
Transfer × Confidence			-0.98 [-2.50, 0.54]	.208
Circle × Rule/Formula × Transfer	0.46 [-0.19, 1.10]	.163	0.41 [-0.45, 1.27]	.349
Circle × Rule/Formula × Confidence			-0.43 [-2.25, 1.38]	.641
Circle × Transfer × Confidence			0.85 [-0.93, 2.63]	.350
Rule/Formula × Transfer × Confidence			-1.26 [-2.77, 0.25]	.101
Circle × Rule/Formula × Transfer × Confidence			-0.49 [-2.09, 1.11]	.549

Table 3.4: Parameters for a cumulative link mixed model using self-reported use of unit circle, self-reported use of rule or formula, and transfer to predict whether a student said they were not at all, somewhat, or very confident their answer to a particular problem was correct.

Predictor	b	p
(Intercept - Not at all | Somewhat)	-4.40*** [-5.63, -3.16]	$<$ .001
(Intercept - Somewhat | Very)	-0.24 [-1.07, 0.59]	.575
Self-reported specific use of unit circle	1.27*** [0.61, 1.93]	$<$ .001
Self-reported specific use of rule / formula	1.88*** [1.15, 2.61]	$<$ .001
Transfer (vs. taught)	-0.13 [-0.47, 0.22]	.467
Circle × Rule/Formula	-0.75† [-1.49, 0.00]	.051
Circle × Transfer	-0.23 [-0.65, 0.18]	.272
Rule/Formula × Transfer	-0.20 [-0.61, 0.22]	.358
Circle × Rule/Formula × Transfer	0.10 [-0.37, 0.58]	.671

as predictors, along with their interactions. Table 3.3 presents the fitted parameters of this model, as well as a model which additionally included confidence as a predictor. There appeared to be a significant interaction of reported use of the unit circle and reported use of a rule or formula, $b = -1.72$, 95% CI $[-3.42, -0.01]$, $z = -1.98$, $p = .048$. On trials in which a student reported not relying on the unit circle, higher rating of rule or formula use significantly predicted better accuracy, $b = 4.10$, 95% CI [0.93, 7.27], $z = 2.54$, $p = .011$. On trials in which a student reported relying a lot on the unit circle, there was no significant effect of self-rated rule or formula use, $b = 0.67$, 95% CI $[-1.13, 2.47]$, $z = 0.73$, $p = .468$. In other words, students who reported using the unit circle or a rule or both on a particular trial were more likely to answer correctly than students who reported relying on neither representation.

After adding confidence along with its interactions with the other predictors included in the model, we found that confidence was a highly significant predictor of accuracy, $b = 3.32$, 95% CI [1.53, 5.12], $z = 3.63$, $p < .001$. However, the interaction of rule use and circle use on accuracy was no longer significant $b = -1.53$, 95% CI [−3.41, 0.36], $z = -1.59$, $p = .112$, along with all other predictors besides confidence. There are, of course, relationships between the representation ratings and confidence. We analyzed these relationships using a cumulative link mixed model, using the self-rated circle use, self-rated rule use, and transfer (vs. taught) to predict confidence, with student and problem type as random factors. Table 3.4 presents the fitted parameters of this model. Higher ratings of circle use predicted higher confidence, $b = 1.27$, 95% CI [0.61, 1.93], $z = 3.76$, $p < .001$. Likewise, higher ratings of rule use predicted higher confidence, $b = 1.27$, 95% CI [0.61, 1.93], $z = 3.76$, $p < .001$. There was a marginal interaction between self-rated use of the unit circle and rule or formula, $b = -0.75$, 95% CI [−1.49, 0.00], $z = -1.95$, $p = .051$. This interaction has a similar story as with accuracy, in which a student is more confident when relying on the circle, a rule, or both, and less confident when relying on neither.

Using a think-aloud protocol to understand the use of rules and the unit circle

To understand the qualitative strategies that students are using, we might look at the four extremes, the corners of Table 3.3. Our think-aloud protocol at the end of the study allows some exploration beyond simple ratings. Note that the experimenter is not aware of the self-reported representation ratings that the student has previously submitted. We will look in particular at cos(20+180) because students showed significant transfer on similar problems when they were not taught in the lesson. We grouped students according to their self-reported use of the unit circle and of a rule or fomula on this problem. Because it was counterbalanced, we have indicated for each student whether this problem was taught or held out for transfer, although we targeted descriptions of transfer, due to our interest in how students generalized their knowledge from the lesson.

First, the group that was neither accurate nor confident were those who reported not using the unit circle and not using a rule. Some students ignored the lesson and tried to use an independently known strategy. Student #8 was fell in this group for problem-specific ratings, but was also the only student in their general ratings to report never using a rule or formula and never any other representation, instead reporting always using the wave graphs.

> STUDENT #8 [transfer]: So for this one, cosine 20 plus 180, I would envision cosine at 180, which goes from 1 to 0 to -1 [index finger moving along wave], and I add 20, so it goes up a little bit, so it's a little bit more than -1 so -0.9 or whatever it is. And then cosine of 20 is 0.9 but positive so I'd go with negative cosine 20.

More commonly, students in this group either completely guessed or made educated guesses based on partial knowledge. Student #26 demonstrated some understanding of rotating 180 degrees, but seemed confused in trying to identify an answer.

> STUDENT #26 [transfer]: So 20 plus 180 [hand pointing right, then flips wrist to left], so I'd do sine minus 20.
>
> EXPERIMENTER: Why is that?
>
> STUDENT #26: I don't know. It feels right.
>
> EXPERIMENTER: It's okay. You were doing some hand motions. You're trying to think of something?
>
> STUDENT #26: Yeah, like a swivel. Like here, then it goes 180 degrees, which is to the other side of the circle.
>
> EXPERIMENTER: Yeah, so you get to that spot on a circle and then you're not sure quite how that maps on to the answers but-
>
> STUDENT #26: No. I just try and visualize the circles somehow. It always goes

clockwise. I don't know.

EXPERIMENTER: No, that's alright.

The second group of students that I will consider consists of those who reported relying a lot on the unit circle and not at all on a rule or formula. Some students, like student #21, described a thorough procedure rooted in the unit circle framework.

> STUDENT #21 [transfer]: And then so cosine 20 plus 180, so I would do the same thing: start at 0. Then I would go to 20 and then I would go all around to 180. So I would like do this in the room too, so I would start here, go to 20, and so 180 would flip it this way. So then I know I'm just evaluating this length here as well. So I know that this length here would be the same as cosine of if I were to kind of flip the circle, like cosine of 20 it would be the same length. But since it's on the other side, it would be a negative value because this is a positive side, this is a negative side, so I'd pick cosine of . . . negative cosine 20.

Other students seemed to be visualizing the unit circle but finding shortcuts or generalizations of this procedure. Student #35 demonstrates some awareness of the sign of trigonometric functions in each quadrant, and applies that general knowledge to the particular case of 20 degrees. This student also avoids comparing lengths of projections by recognizing the function only switches in this task with a shift of ±90 degrees.

> STUDENT #35 [transfer]: So, now I draw a 20 degree angle, at 180 so it flips to the third quadrant of the unit circle, and basically when you go to the third, everything is negative, so I know that the answer has to have a negative sign in it. And also because this is the 180 flip, cosine stays the same as cosine, so I choose negative cosine 20

A third group of students reported relying a lot on the unit circle and also a lot on a rule or formula in order to solve $\cos(20+180)$. Student #29 described using a rule that summarized the symmetric relationship observed in the unit circle.

> STUDENT #29 [taught]: Okay so, when there's 180 it just kind of... flipity flops it, no matter if it's sine or cosine. So cosine of 20 plus 180 is the same as negative cosine of 20.
>
> EXPERIMENTER: Okay, so for that one you are using the "flipity flopity" rule.
>
> STUDENT #29: Yeah.
>
> EXPERIMENTER: So you don't need to really visualize anything for that one.
>
> STUDENT #29: Um, I guess I still kind of think of rules and steps on some of them, but I still always pop it up with that unit circle, so here (horizontal hand on right) and then flipity flopity it (moves hand to left).
>
> EXPERIMENTER: So you know that it's on the other side of the circle.
>
> STUDENT #29: Yeah.
>
> EXPERIMENTER: And so that helps for you to remember maybe partly the flipity flopity part.
>
> STUDENT #29: Yeah, exactly.

Other students described using rules on certain problem types, where they perhaps were more confident. Student #58 described their general strategy as sometimes automatic and sometimes circle-based: "Anything +0 or +/- 180 was automatic, didn't need to visualize for those. For expressions involving x +/- 90, I would visualize two angles on the plane. ..." This student generalized the relationship shown in the lesson to plus or minus 180, and perhaps struggled to find similar generalizations for shifts involving plus or minus 90.

> STUDENT #58 [transfer]: cos(20 + 180)... so it's negative cosine 20 and I choose that because... I just know that plus or minus 180 flips the [hand over hand motion] sign [fingers switching motion]. So theta, if you have cosine or sine theta plus or minus 180, then it's gonna be negative cosine or sine.
>
> EXPERIMENTER: And that's again ...
>
> STUDENT #58: [crosstalk] another rule ...
>
> EXPERIMENTER: [crosstalk] the rule from the last time, or maybe even before, and you're not really visualizing anything to do that?
>
> STUDENT #58: [crosstalk] No, it's just like ...
>
> EXPERIMENTER: [crosstalk] You just know to flip the sign.
>
> STUDENT #58: Yeah.

The final and perhaps most interesting group of students reported relying a lot on a rule or formula but not at all on the unit circle. Student #25 fell in this group for problem-specific ratings, but was also the only student in their general ratings to report always using a rule or formula and never any other representation. Their written description of general strategy was brief and dry: "I applied the knowledge of relations to 180 90 and 0 that were provided to me in the previous steps." However, in their think-aloud protocol, they expressed the use of a rule, backed by the ability to visualize the unit circle.

> STUDENT #25 [transfer]: Okay, so now, it's going to be 20 plus [180], so I know it's going to go basically shift right across that [rotates hand], a straight line across the axis, and then it said that when you have [180] plus 20, it's just pretty much negative of whatever the other angle is, so it's going to be negative cosine of 20.

EXPERIMENTER: Alright, and again, you can answer pretty quickly using that rule, but you know how to visualize?

STUDENT #25: Yeah.

Problem-specific behavior on $\cos(-\theta + 0)$

With the problem-specific reports collected in this study, we can ask some final questions about a problem that, although always taught in the lessons, may illuminate the strategies students are using after the grounded lesson. In our previous studies, use of the unit circle was strongly associated with successful performance on $\cos(-\theta + 0)$ problems. Participants who did not rely on the unit circle suffered on this seemingly simple problem, typically choosing the response $-\cos(\theta)$. One potential explanation for their struggle is a simple heuristic strategy which could be described as 'pulling out the minus sign' – something that is consistent with symbolic rules in some situations (e.g. $(-A) = -(A)$), and happens to work for $\sin(-\theta+0)$. In Figure 3.10, behavior on this particular problem is examined, using data from this study and from our previous study, in which we collected problem-specific reports for the group which received no lesson.

Of course, we expect increased accuracy after a lesson, so we restricted the data to trials which the student answered correctly. In order to compare the same kind of problems, we also restricted data from this study to those problems with a specific angle, excluding those problems with a generic θ. Using a cumulative link model to predict each rating, we found no significant difference in self-reported circle use between those in the current study's grounded lesson and those who received no lesson, $b = -0.16$, 95% CI $[-0.95, 0.63]$, $z = -0.40$, $p = .693$. There was also no significant difference in self-reported confidence, $b = 0.52$, 95% CI $[-0.63, 1.74]$, $z = 0.89$, $p = .375$. However, students in the current study's grounded lesson did report significantly greater reliance on a rule or formula, $b = 1.09$, 95% CI $[0.28, 1.95]$, $z = 2.59$, $p = .010$. In an independent t-test using log-transformed response times, students in the grounded lesson solved the problem in significantly less

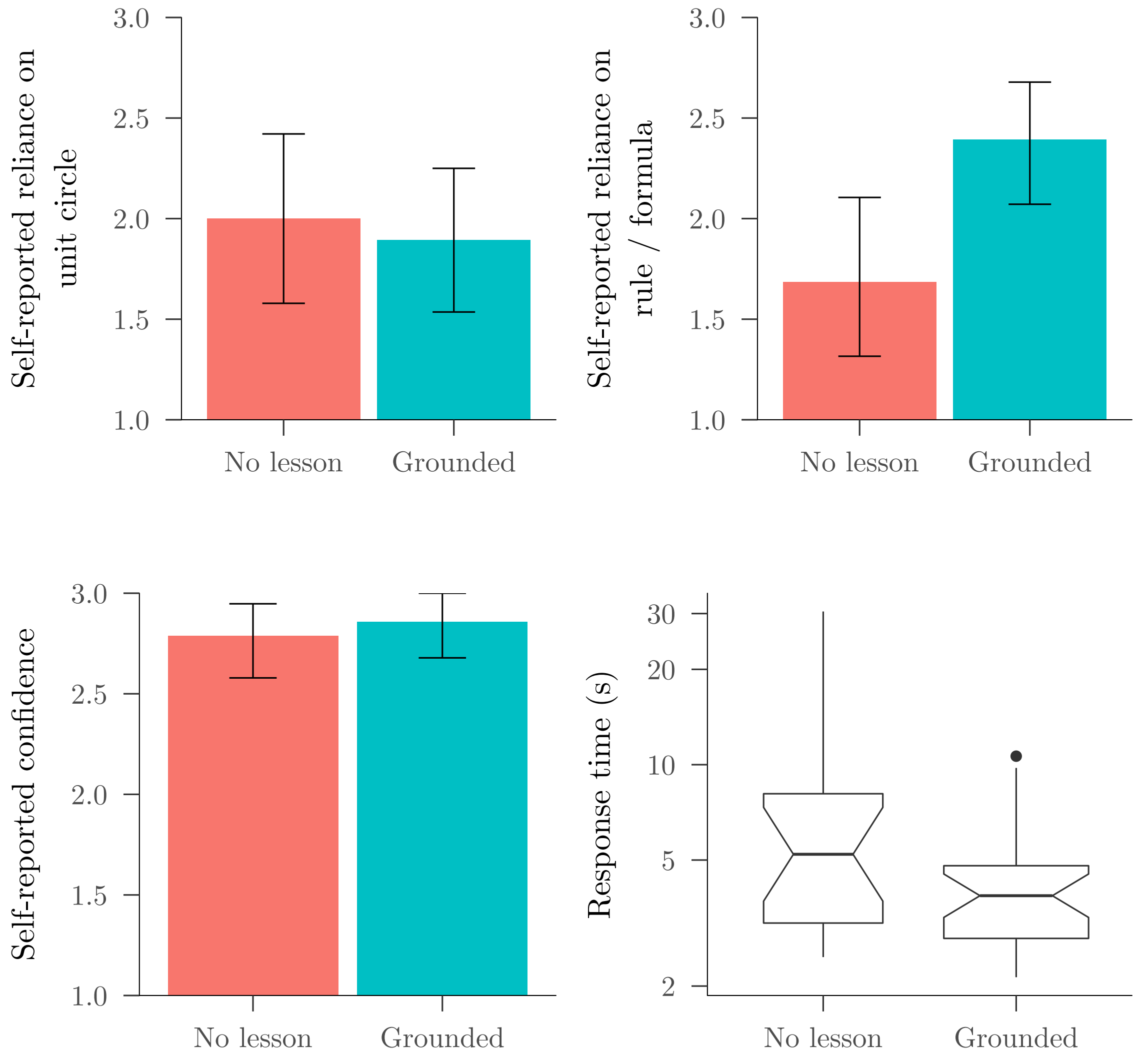


Figure 3.10: Problem-specific ratings of circle use, rule / formula use, confidence and response time for solving $\cos(-\theta + 0)$ on correct trials only, comparing students who saw the grounded lesson and those with no lesson

time, $t(45) = -2.09$, $p = .042$.

3.2 Discussion

This chapter explored the pattern of representation use in solving trigonometric identities, and in particular, we sought to understand how a lesson grounded in the unit circle facilitates success on transfer problems. One potential explanation was that students learned that the unit circle could be applied on every problem, and used a quantitative interpretation of each expression. However, students who saw the grounded lesson reported increased use of both rules and the unit circle. This suggests that rules do play an important role in problem solving, and that students were not simply relying on a universal unit circle procedure. Students were also less likely to report using the unit circle when the problem contained a variable, instead of a specific value that is associated with a specific angle on the unit circle. A different explanation of transfer after the grounded lesson was that students were utilizing both the unit circle and rules as independent strategies, where they could use rules on taught problems, and use the unit circle on problems for which they cannot recall a rule. Indeed, we found that students reported greater use of rules on taught problems than on transfer problems. The grounded lesson even facilitated successful rule use on $\cos(-\theta + 0)$, which students with no lesson often answered incorrectly due to a faulty rule. Note though that, of students who reported relying a little or a lot on the unit circle or on a rule or formula, 42%, 95% BCI [34, 51], reported relying a little or a lot on the other representation during the same trial. With this many students relying on both representations during the same trial (for taught or transfer problems), we should consider more nuanced descriptions of the relationship between rules and the unit circle. Furthermore, a student who, after the grounded lesson, reports using only a rule to solve a problem, may be using a very different rule than the one learned by students in the formal lesson, without the unit circle. Specifically, the grounded lesson may facilitate students encoding and expressing rules in terms of the unit circle, which may in turn support accuracy and generalization. The think-aloud protocols provided initial (exploratory) support for this, finding students who reported using rules and the unit circle or even rules alone that grounded their rules in terms of the unit circle, its angles, and projections onto the XY coordinate plane. While further work is needed to tease apart the exact nature of a rule grounded in the unit circle, Study 3 reinforces the idea that a grounded understanding of trigonometric relationships facilitates transfer to novel problems, and suggests that students with such an understanding rely on both rules and the unit circle for success across all kinds of problems.

Chapter 4

Readiness for Successful Reasoning with Grounded Rules

If rules grounded in the unit circle indeed help students to solve trigonometry problems and generalize to other problems, both educators and cognitive scientists should seek to understand the process of learning and mastering these grounded rules. In this chapter, I will first consider how to identify students who are ready to learn and use the unit circle successfully to solve trigonometric identities. Then, I will explore how students who have mastered the unit circle may reduce its active role in problem solving while maintaining rules grounded in the unit circle as a coherent conceptual structure.

4.1 Readiness for Grounding Rules in the Unit Circle

Resnick (1983) proclaimed that "all learning depends on prior knowledge." In fact, we may consider learning not as an addition of isolated facts, but as a growth or change of conceptual relationships.

This view is fundamental to many perspectives in cognitive psychology. Constructivism, from the individually guided (Inhelder & Piaget, 1958) to the socially and culturally situated (Vygotsky, 1962), emphasizes a learner forming new concepts and making sense of an experience in terms of existing concepts and schemata (Roschelle, 1995). From a connectionist perspective, the knowledge that guides processing is implicit in the strength of connections between simple units, and learning occurs by changing the strength of those connections (Rogers & McClelland, 2014; Rumelhart, Hinton, & McClelland, 1986). A Bayesian perspective (or more generally, a computational level of analysis) balances the integration of new information with the strength of existing beliefs (Perfors, Tenenbaum, Griffiths, & Xu, 2011). Understanding the strength and nature of prior knowledge is essential to interpreting learning effects and to tailoring developmentally appropriate lesson materials.

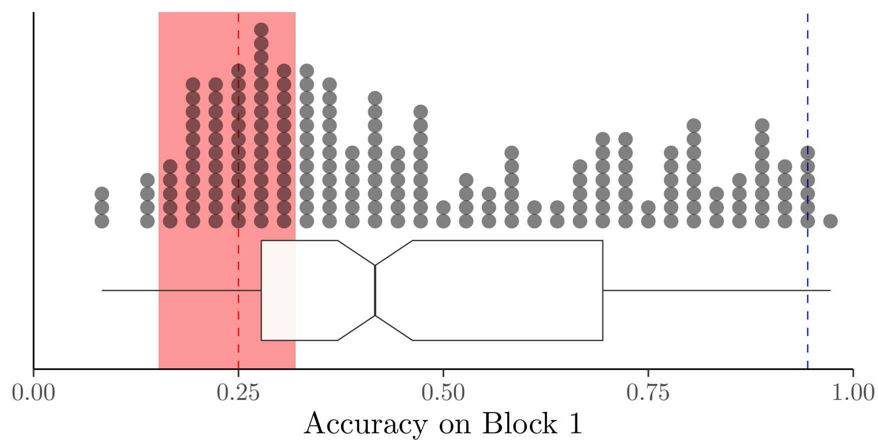


Figure 4.1: Distribution of each student's mean accuracy on problems in Block 1. The red region highlights the range of values that are not significantly different from chance (25%, at the red dashed line). The blue dashed line indicates a ceiling threshold, for those who answered two or fewer problems incorrectly.

Across all our studies (including the pilot study), we observed a wide range of performance on block 1, including a number of students within range of chance, as Figure 4.1 shows. There is a wide variety of factors that contribute to this variability. While there are likely additional sources of readiness that are not captured, I believe that the problems on block 1 provide a good opportunity to see students demonstrate their prior knowledge and its relevance to the trigonometric identity problems featured in our studies. (Note that these problems may be less helpful measures in other populations, particularly when scores are more concentrated around the level of chance.)

Because certain concepts, skills and attitudes seem necessary (or at least helpful) in order to understand the grounded lesson, we expect those near chance to benefit less from the grounded lesson than students above chance (excluding those near ceiling). Indeed, in Studies 1 and 2, it appeared that students near chance show some benefit from the grounded lesson, but somewhat less than the improvement in performance by students who were above chance (but below ceiling) on the first block. However, Study 3 did not follow this pattern. In Figure 4.2, we see that the mean improvement of students near chance was actually slightly higher than the mean improvement of students above chance.

To examine the effects of block 1 performance (near versus above chance) and lesson condition, we used a linear regression to predict improvement from block 1 to block 2. This analysis included data from Studies 1, 2, and 3, as well as the pilot study (excluding students in the wave-based lesson), in order to achieve higher power. The fitted parameters of this model are shown in Table 4.1. There was no significant difference in improvement between those who were near chance in block 1 and those who were above chance, $b = 0.00$, 95% CI [−0.03, 0.02], $t(192) = -0.08$, $p = .937$. There was no significant interaction between block 1 performance category and general lesson effect (both lessons vs. no lesson), $b = -0.01$, 95% CI [−0.02, 0.01], $t(192) = -0.84$, $p = .404$, or between block 1 performance category and lesson type (grounded lesson vs. formal lesson), $b = 0.01$, 95% CI [−0.02, 0.04], $t(192) = 0.36$, $p = .722$. There was also no significant simple effect of block 1 performance category within students who saw the grounded lesson, $b = 0.01$, 95% CI [−0.06, 0.07], $t(192) = 0.16$, $p = .870$. We then added transfer (vs. taught) as an additional predictor along

Table 4.1: Fitted model parameters using block 1 performance, lesson condition, and, for one model, transfer (vs. taught) to predict block 2 performance.

	Simple model		Model including transfer	
Predictor	*b*	*p*	*b*	*p*
(Intercept)	0.12*** [0.10, 0.15]	$<$ .001	0.13*** [0.11, 0.16]	$<$ .001
Block 1 (above chance vs. near chance)	0.00 [-0.03, 0.02]	.937	0.00 [-0.03, 0.02]	.767
Lesson (vs. none)	0.02* [0.00, 0.04]	.021	0.03** [0.01, 0.04]	.003
Lesson type (grounded vs. formal)	0.03† [0.00, 0.06]	.079	0.02 [-0.01, 0.05]	.196
Transfer (vs. taught)			-0.03*** [-0.05, -0.01]	$<$ .001
Block 1 × Lesson	-0.01 [-0.02, 0.01]	.404	-0.01 [-0.03, 0.01]	.272
Block 1 × Lesson type	0.01 [-0.02, 0.04]	.722	0.01 [-0.02, 0.03]	.719
Block 1 × Transfer			0.01 [-0.01, 0.02]	.346
Lesson × Transfer			-0.02** [-0.03, 0.00]	.010
Lesson type × Transfer			0.02* [0.00, 0.04]	.037
Block 1 × Lesson × Transfer			0.01 [-0.01, 0.02]	.293
Block 1 × Lesson type × Transfer			0.00 [-0.02, 0.02]	.995

with its interactions with block 1 performance category and lesson condition and their three-way interaction. As a result of this second data point per student, we used a mixed model with subject as a random effect. The three-way interaction between block 1 performance category, general lesson effect, and transfer was not significant, $b = 0.01$, 95% CI $[-0.01, 0.02]$, $t(192) = 1.05$, $p = .293$. The three-way interaction between block 1 performance category, lesson type, and transfer was also not significant, $b = 0.00$, 95% CI $[-0.02, 0.02]$, $t(192) = 0.01$, $p = .995$. Within students who saw the grounded lesson, there was no significant interaction of block 1 performance category and transfer, $b = 0.01$, 95% CI $[-0.01, 0.04]$, $t(192) = 1.20$, $p = .230$. There was also no significant simple effect of block 1 performance category within students who saw the grounded lesson, $b = -0.01$, 95% CI $[-0.04, 0.03]$, $t(192) = -0.45$, $p = .655$.

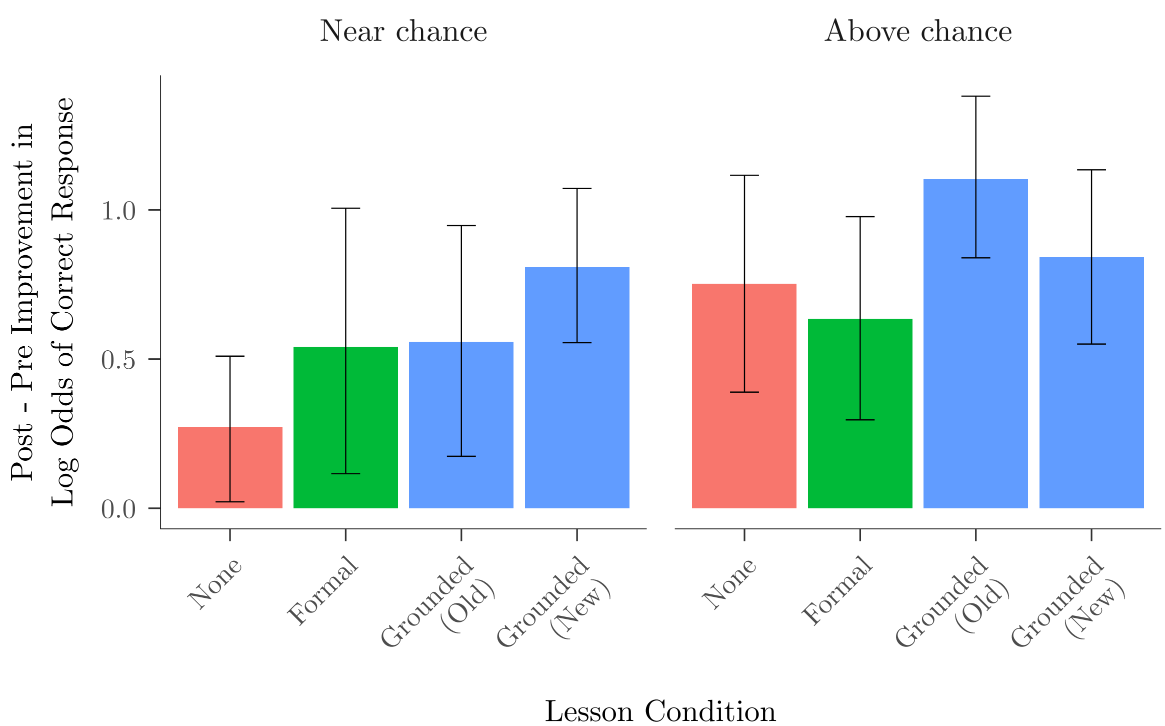


Figure 4.2: Mean improvement in log odds of responding correctly (with 95% BCIs) for students whose accuracy on Block 1 was near chance and for those whose accuracy on Block 1 was above chance but below ceiling, split by lesson condition.

There are a few issues with this initial consideration of readiness to learn from the grounded lesson. First, the accuracy on block 1 is a practically continuous measure (with scores ranging from 0 to 40 out of 40 problems), but we have split students into two groups. (More specifically, we split students into four groups: those below chance, those near chance, those above chance but below ceiling, and those near ceiling. We then excluded the lowest and highest groups as extremes with few students.) Second, the dependent variable was the difference between a student's mean block 2 performance and mean block 1 performance. If we treat block 1 accuracy as a continuous predictor, we can use it to predict accuracy on each trial in block 2. (With two split groups, we predicted difference scores rather than block 2 performance in order to avoid losing information about each student's starting point.) A trial-level analysis allows us to model more accurately each student's performance.

We used a logistic mixed model to predict whether a student answered each problem correctly in block 2. We were primarily interested in the potential predictive power of the three-way interaction between lesson condition, transfer (vs. taught), and block 1 performance, with linear, quadratic and cubic terms included to capture the non-linearities across block 1 performance. We also included an additive predictor for gender (for the fully interactive model, a maximum likelihood approach failed, and using Bayesian MCMC methods, we did not find any improvement in WAIC over the additive model). We included a random intercept and random effect of transfer for each subject, as well as a random intercept for each problem type. This analysis includes data from Studies 1, 2, and 3 in order to have stronger evidence across the range of block 1 performance[1].

The fitted parameters of this model are shown in Table 4.2. There were no significant interactions involving block 1 performance, looking across its linear, quadratic and cubic contrasts. There was no significant three-way interaction between block 1 performance, lesson condition, and transfer,

[1]Data from the pilot study was not included because gender data was not collected.

Table 4.2: Parameters for a logistic mixed model using lesson condition, transfer (vs. taught), and block 1 performance (with linear, quadratic and cubic terms) to predict whether each trial was answered correctly.

Predictor	b	95% CI	p
(Intercept)	0.65***	[0.33, 0.97]	$< .001$
Block 1 performance (linear)	87.80***	[72.29, 103.32]	$< .001$
Block 1 performance (quadratic)	18.50**	[5.65, 31.34]	.005
Block 1 performance (cubic)	-21.16**	[-34.99, -7.33]	.003
Lesson (vs. none)	0.18**	[0.07, 0.30]	.002
Lesson type (grounded vs. formal)	0.14	[-0.06, 0.34]	.173
Transfer (vs. taught)	-0.20*	[-0.35, -0.05]	.011
Gender (female vs. male)	-0.23**	[-0.37, -0.08]	.003
Block 1 (linear) × Lesson	-1.27	[-11.21, 8.67]	.802
Block 1 (quadratic) × Lesson	3.41	[-6.31, 13.13]	.491
Block 1 (cubic) × Lesson	6.27	[-4.44, 16.98]	.251
Block 1 (linear) × Lesson type	-12.62	[-31.42, 6.17]	.188
Block 1 (quadratic) × Lesson type	-4.66	[-20.63, 11.32]	.568
Block 1 (cubic) × Lesson type	-8.44	[-22.54, 5.67]	.241
Block 1 (linear) × Transfer	-8.56	[-20.18, 3.07]	.149
Block 1 (quadratic) × Transfer	-6.73	[-18.77, 5.31]	.274
Block 1 (cubic) × Transfer	-6.68	[-19.92, 6.57]	.323
Lesson × Transfer	-0.14***	[-0.22, -0.07]	$< .001$
Lesson type × Transfer	0.22**	[0.09, 0.36]	.001
Block 1 (linear) × Lesson × Transfer	4.48	[-4.12, 13.08]	.307
Block 1 (quadratic) × Lesson × Transfer	-4.28	[-13.18, 4.62]	.346
Block 1 (cubic) × Lesson × Transfer	5.74	[-5.09, 16.57]	.299
Block 1 (linear) × Lesson type × Transfer	7.54	[-7.03, 22.11]	.311
Block 1 (quadratic) × Lesson type × Transfer	2.01	[-12.91, 16.93]	.792
Block 1 (cubic) × Lesson type × Transfer	5.95	[-7.48, 19.38]	.385

$\chi^2(6) = 4.55$, $p = .603$. There was also no significant two-way interaction between block 1 performance and lesson condition, $\chi^2(6) = 4.81$, $p = .569$, or between block 1 performance and transfer, $\chi^2(3) = 2.68$, $p = .444$.

Block 1 performance had a significant cubic main effect on block 2 performance, $b = -21.16$, 95% CI [-34.99, -7.33], $z = -3.00$, $p = .003$, a significant quadratic main effect, $b = 18.50$, 95% CI [5.65, 31.34], $z = 2.82$, $p = .005$, and a significant linear main effect, $b = 87.80$, 95% CI [72.29, 103.32], $z = 11.09$, $p < .001$. To understand these effects, Figure 4.3 shows the pattern of improvement on taught problems and on transfer problems for each lesson condition as a function of block 1 performance. Although block 1 performance and its effect on block 2 performance are continuous, we can interpret the linear, quadratic, and cubic effects by considering four groups of students. First, students who are near ceiling are limited in their potential improvement. The second group of students, those who performed above chance on block 1 but below ceiling, includes those who might benefit the most from a lesson. Third, students who performed near chance on block 1 seem to benefit less from a lesson (particularly transfer problems). Finally, the small group of students who performed below chance may have been mistaken but thoughtfully engaged with the problems, and so after a lesson, or even after self-reflection without a lesson, they can show strong improvement. The differences between taught and transfer problems are striking. On taught problems, both the grounded lesson and formal lesson show strong improvement relative to no lesson. This effect holds across different levels of block 1 performance, although students with high block 1 performance are limited by the ceiling of the test. On transfer problems, students in the grounded lesson show greater improvement than students in the formal lesson or those with no lesson. Although none of the interactions with block 1 performance were significant, it appears that, relative to no lesson, the grounded lesson helps most those in the second quartile (between 25% and 50%), while the formal lesson appears to be hindering or interfering the transfer performance for students who performed very well on block 1.

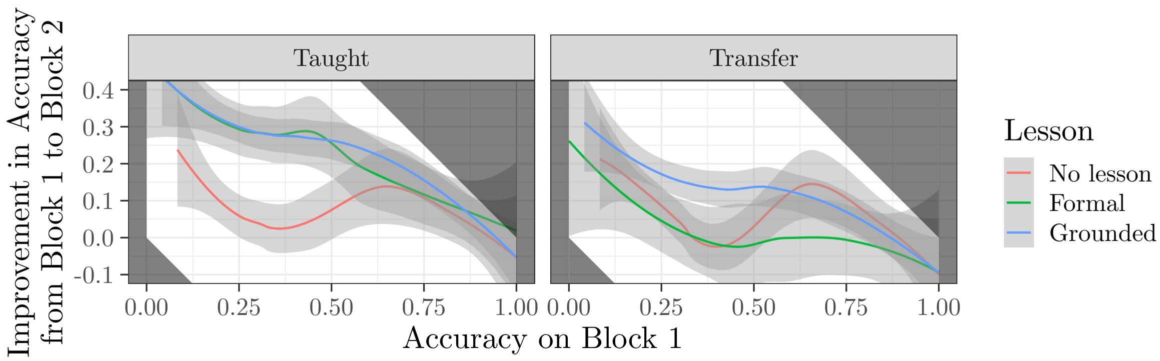


Figure 4.3: Relationship between accuracy on block 1, and improvement in accuracy on block 2, for each lesson condition, broken down by taught vs. transfer. The lines shown are LOESS curves fit to the data and their 95% confidence bands.

4.2 Widening the Range of Prior Knowledge and the Nature of Learning

4.2.1 High school pilot

The effect of prior knowledge in our Stanford studies may be difficult to interpret due to the varying durations of time since last encountering trigonometry. That is, a student with strong prior experience may perform poorly on block 1, but may be refreshed by the lesson through recognition and cued recall. In order to understand the effect of differences in prior knowledge while that knowledge has been formed or used recently, we conducted a pilot study with high school students. We recruited juniors and seniors who were enrolled in a regular (non-honors) track precalculus class at Palo Alto High School. They had been exposed to several different representations (the right triangle, the unit circle, the waves, a few rules / formulae, and a mnemonic or two) that could help them solve the problems in our task.

Students completed the study on laptops in class, and participated in exchange for extra credit. Due to the time constraint of their class period, each block consisted of 20 problems (rather than the 40 problems Stanford students saw). All students were shown the grounded lesson.

16 of the 17 participants performed near chance on the pre-test, as shown in Figure 4.4. While the sample was restricted both in range of performance and in size, we asked whether the grounded lesson helped improve performance from block 1 to block 2 using a logistic mixed model. In the model, we used block, transfer (vs. taught), and their interaction to predict whether each trial was answered correctly. We included a random intercept for each participant, as well as a random effect of block, transfer, and their interaction. The interaction of block and transfer was not significant, $b = -0.06$, 95% CI [-0.25, 0.13], $z = -0.65$, $p = .517$, and neither was the effect of transfer, $b = -0.13$, 95% CI [-0.33, 0.07], $z = -1.31$, $p = .191$. Students did show marginally significant improvement from block 1 to block 2, $b = 0.20$, 95% CI [0.00, 0.40], $z = 1.92$, $p = .055$. Interpretation of these results is limited due to the low power resulting from a small sample of students, most of whom had limited prior experience.

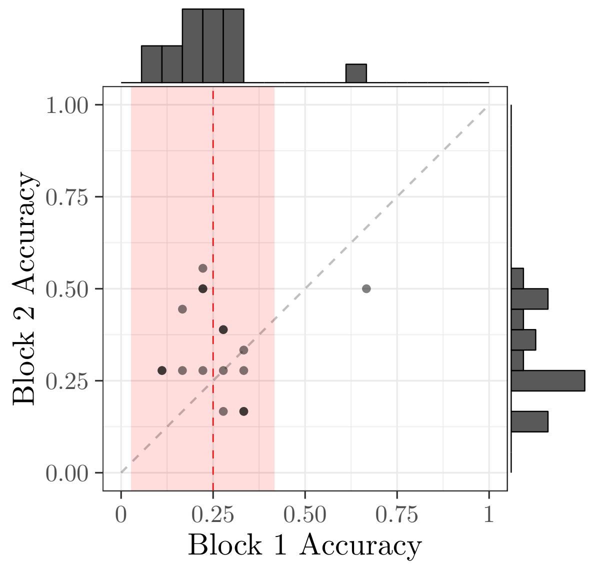


Figure 4.4: The distribution of each student's mean accuracy on Block 1 and on Block 2 from our high school pilot study. Chance is shown as a red dashed line at 25%, and points within the red shaded area represent students whose block 1 accuracy was not significantly different from chance.

4.2.2 Trig Academy

Our high school pilot study highlighted two areas for improvement: measurement of the prior knowledge of students in more meaningful ways, and development of the grounded lesson as a stronger and more engaging learning experience. To be reported fully elsewhere, we developed Trig Academy, an online interactive learning platform that addresses both of these issues.

Students, recruited both from local high school classes and from a community college research pool, completed a pre-test, a lesson spread over six sessions, and a post-test. The pre-test consisted of several different parts. We included 10 multiple choice identity problems, like those in our Stanford studies. The probe expressions for these problems were constructed by crossing the function (sine or cosine), the sign of the angle θ (positive or negative), and the absolute magnitude of the angle

Δ (0, 90, or 180). The sign of Δ (positive or negative) and order (θ before Δ or Δ before θ) were randomly determined. However, we also had a wide variety of other tasks to collect potentially relevant measures of a student's attitudes, abilties, and knowledge. We included aptitude tests of mental rotation, hidden figure verification, and general pattern recognition. We also included tests of mathematical skills like marking points on a plane, marking angles on a circle and identifying sine or cosine as ratios of the sides of a triangle. These tests were conducted twice, once without any external support, and once with basic cues to support student recall and performance. Finally, we collected attitudinal measures by asking students (either in a pre-screen session or in the pre-test) to rate statements about the nature of mathematics, the ease of spatial thinking, and growth versus fixed mindsets.

The unit circle integrates two fundamental representations: the X-Y coordinate plane, and angles as rotations around a circle. Therefore, in the lesson materials for Trig Academy, we walk participants from marking a number line to marking angles around a circle, before introducing sine and cosine. We hypothesize that if students have firm knowledge of these components, they will be able to learn and use the unit circle after progressing through the Trig Academy.

4.3 Transitioning Away from Explicit Use of the Unit Circle

4.3.1 Reducing self-reported use of the unit circle

A student who has mastered the unit circle may not need to be actively using it, in order to succeed at solving trigonometric identity problems. Here, we will explore how students may reduce their active use of the unit circle as they progress and gain more experience and confidence. We can ask whether students changed their strategy over the course of the problem-specific self-report block in Studies 1 and 3. This block consisted of 20 problems.

Figure 4.5 shows the pattern of self-reported reliance on the unit circle and on a rule or formula over the course of 20 trials. Students rated their reliance on different representations on a three-point scale (“None”,“A little”, “A lot”), so we used a cumulative link model. This model was multivariate to allow us to predict the ratings of both reliance on the unit circle and reliance on a rule or formula. In order to fit such a model, we used a Bayesian approach (Burkner, 2017), with four Monte Carlo Markov chains, each of 10,000 iterations (including half as warmup). We used uninformed or very weakly informed prior distributions for all parameters. We included trial number, theta, and condition as predictors, as well as the interaction of trial and theta and the interaction of trial and lesson condition. (Note that we did not include an interaction between theta and lesson condition because theta is actually nested within lesson condition, as the participants in Study 1 saw only problems with specific angles.) We also included a random intercept and effect of trial for each student, and a random intercept and effect of theta for each problem type. Because Study 1 contained no lesson, we included only correct trials (from both studies) in this analysis, in order to increase the relative frequency of trials in which a student exerted legitimate effort (as opposed to guessing or employing irrelevant strategies). After fitting the model, none of the parameters had a Monte Carlo standard error greater than 10% of the posterior standard deviation, and none had a $\hat{R}$ value above 1.01. Trace plots appeared to show chains overlapping and mixing well.

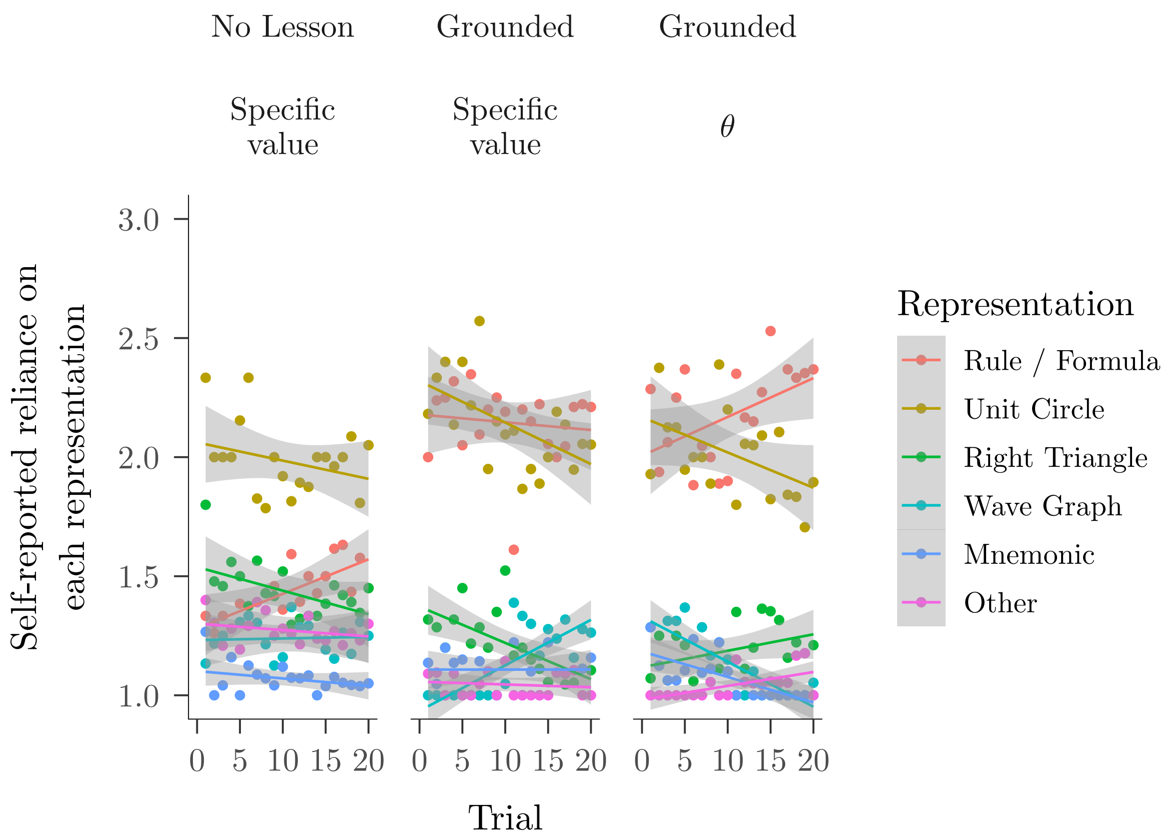


Figure 4.5: Students' self-reported strategy over the course of 20 trials, broken down by lesson condition and theta type. Students rated the extent to which they relied on a rule / formula and on the unit circle (and on other representations, not included) on a 3-point scale ('None','A little', 'A lot'). Each point is a mean rating across trials which students answered correcly.

The fixed effect parameters are shown in Table 4.3. On later trials, students reported less reliance on the unit circle than on earlier trials, $b = -0.06$, 95% CI $[-0.12, -0.01]$, $n_{\text{Eff}} = 9,335$. (Note that in this context, 95% CI refers to a Bayesian credible interval). Relative to no lesson, the grounded lesson caused marginally higher self-reported use of the unit circle, $b = 0.89$, 95% CI $[-0.01, 1.81]$, $n_{\text{Eff}} = 1,800$, and much higher self-reported use of a rule or formula, $b = 2.06$, 95% CI $[1.28, 2.89]$, $n_{\text{Eff}} = 2,704$. When students were shown a problem involving θ instead of some specific angle, they reported less reliance on the unit circle, $b = -0.34$, 95% CI $[-0.59, -0.09]$, $n_{\text{Eff}} = 20,000$. There was

Table 4.3: Parameters (with effective sample sizes) for a logistic mixed model using lesson condition, transfer (vs. taught), and block 1 performance (with linear, quadratic and cubic terms) to predict whether each trial was answered correctly.

	For circle:		For rule / formula:	
Predictor	b	n_{Eff}	b	n_{Eff}
(Intercept - Not at all | Somewhat)	-0.18 [-1.10, 0.78]	2,000	0.29 [-0.58, 1.18]	2,676
(Intercept - Somewhat | Very)	1.98 [1.05, 2.96]	2,015	2.74 [1.86, 3.67]	2,736
Lesson (grounded vs. none)	0.89 [-0.01, 1.81]	1,800	2.06 [1.28, 2.89]	2,704
Trial	-0.06 [-0.12, -0.01]	9,335	0.00 [-0.07, 0.07]	6,899
Theta (vs. instantiated)	-0.34 [-0.59, -0.09]	20,000	0.06 [-0.24, 0.36]	13,967
Lesson × Trial	-0.01 [-0.07, 0.04]	8,805	0.03 [-0.04, 0.10]	6,411
Trial × Theta	0.00 [-0.07, 0.06]	8,371	-0.03 [-0.10, 0.03]	9,217

no significant interaction between trial and condition for circle ratings, $b = -0.01$, 95% CI [-0.07, 0.04], $n_{\text{Eff}} = 8,805$, or for rule/formula ratings, $b = 0.03$, 95% CI [-0.04, 0.10], $n_{\text{Eff}} = 6,411$. There was also no significant interaction between trial and theta for circle ratings, $b = 0.00$, 95% CI [-0.07, 0.06], $n_{\text{Eff}} = 8,371$, or for rule/formula ratings, $b = -0.03$, 95% CI [-0.10, 0.03], $n_{\text{Eff}} = 9,217$.

4.3.2 Reducing use of external unit circle tool

While our Stanford studies relied on self-reports in order to determine a student's strategy, our Trig Academy studies allowed us to monitor manipulation of external representations. In the post-test, students saw 40 identity problems, constructed in a similar way to our Stanford studies. On the first 20 problems, students were provided an interactive unit circle on the same page as each problem, as shown in Figure 4.6. They could click anywhere to place a point, and the point would appear along the circumference of the circle, along with a ray (but no projections). Students could click again to relocate the point, or drag it around the circle. The circle had the four cardinal angles labelled (0, 90, 180, and 270 degrees). Tick marks were at each tenth of a unit along the axes and at each 10 degrees around the circle.

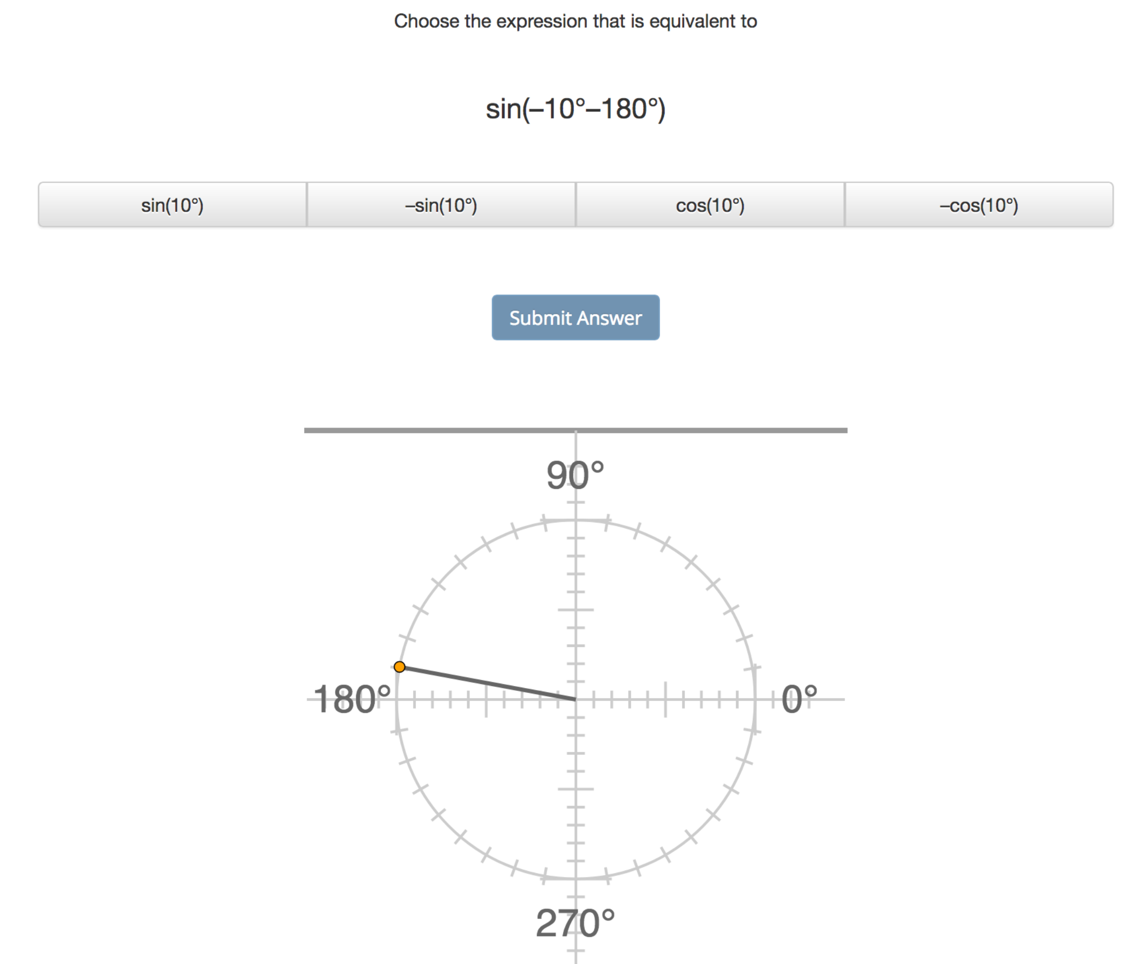


Figure 4.6: The unit circle tool available for students to use in the post-test of our Trig Academy studies. Here, a student has already clicked to place a ray at 170 degrees.

First, we can look at how often these interactions occurred over the course of 20 problems. If the pattern matches the self-reported circle use by Stanford students, students should be using the external circle less often as they progress through the test. However, our previous finding could be influenced by self-report tendencies. For instance, as students progressed to later trials, they may have decided to focus their self-reported reliance to only one representation (although allowed to report reliance on more than one representation). To predict whether or not a student actually interacted with an external unit circle, we use a logistic mixed model. We included trial number as a predictor, while also allowing its effect and a random intercept to vary for each subject and for each subject pool (high school and community college). Indeed, trial had a significant effect on use of the external circle, $b = -0.12$, 95% CI $[-0.20, -0.05]$, $z = -3.26$, $p = .001$. As students progressed through the test, they were less likely to interact with the external circle. Students could still be actively manipulating a mental representation of the unit circle. However, this finding corroborates our previous finding, in which Stanford students reported less circle use as they progressed to later problems, and ensures our claims are not based only on students' self-report tendencies.

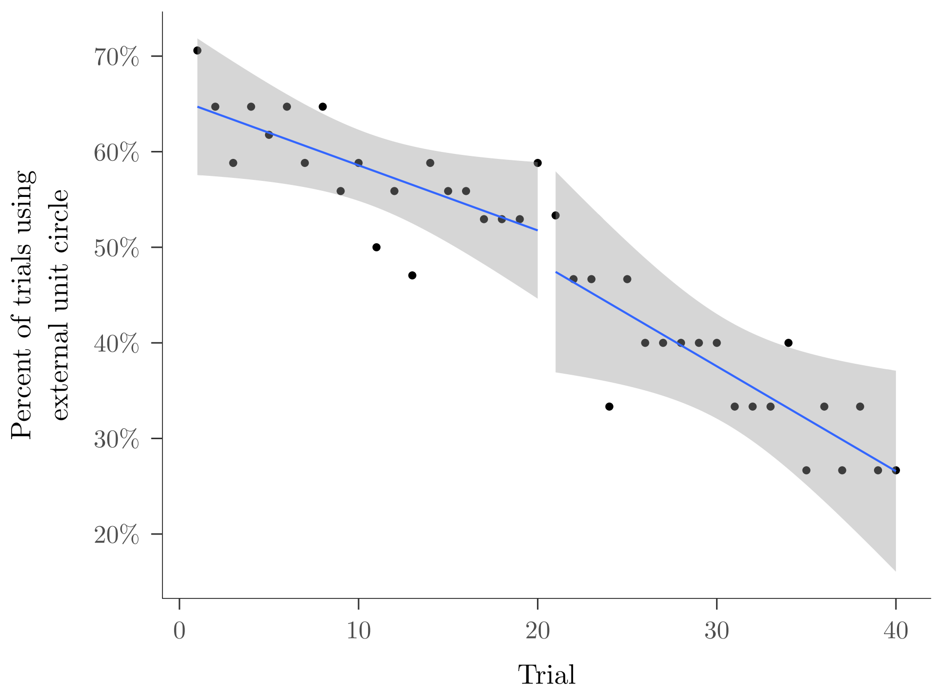


Figure 4.7: Percent of trials using the external unit circle tool as a function of trial number, progressing through the session. Each point represents the average across students for each trial.

Within trials in which the student interacted with the external circle, trials varied in the number of times a student clicked or dragged a point. To predict how many times a student interacted with an external unit circle on a trial, we use a negative binomial mixed model, with only trials where a student interacted at least once with the external circle. As before, we included trial number as a predictor, while also allowing its effect and a random intercept to vary for each subject and for each subject pool (high school and community college). Trial had a significant effect on use of the external circle, $b = -0.02$, 95% CI $[-0.04, -0.01]$, $z = -3.44$, $p = .001$. As students progressed through the test, they were making fewer clicks and drags on trials where they interacted with the unit circle. This trend is shown in Figure 4.7. This effect may partially be due to general behavioral optimization and effort reduction while using the circle (e.g., reducing misclicks and corrections). Students did though use a similar interactive diagram through extensive lesson materials. They could be transitioning to more mental manipulation of the the unit circle. The circle tool only displayed one point at a time, so if a student learned to imagine a second point rather than clicking the second point, they could compare the properties (i.e., sine and cosine) of the displayed point with those properties of their imagined point. Finally, students could be trusting their beliefs in relationships without needing external verification. For instance, they might be confident that their response should have the opposite function to an expression with a 90 degree shift, but they use the external circle to determine the sign of the expression. With the knowledge that all response choices have an angle in the first quadrant, there would then be no need to evaluate the quantitative value of each response choice.

Finally, we can consider the relationship between external circle use and accuracy. We used a logistic mixed model, with subject as a random factor, to predict whether a student answered each trial correctly. To avoid undue influence of outliers where a student happened to click many times on a single trial, we defined circle use as a binary variable: whether or not a student had any interaction with the external circle during each trial. We can distinguish the between-subject effect and the within-subject effect by using two different predictors in our model (Curran & Bauer, 2010): the student's mean circle use, and the student's circle use on a particular trial minus their mean circle use. We then also included the within-subject effect as a random effect allowed to vary among students. Circle use was highly predictive of accuracy when characterized as a between-subject effect, $b = 1.89$, 95% CI [0.92, 2.86], $z = 3.83$, $p < .001$. Students who interacted with the external circle on more trials tended to have higher overall accuracy. However, the within-subject effect of circle use did not have a significant effect on accuracy, $b = 0.29$, 95% CI [-0.33, 0.90], $z = 0.92$, $p = .359$. Whether or not a student happened to interact with the external circle on a particular trial did not predict whether a student was more likely to answer that problem correctly.

4.4 Discussion

In this chapter, we explored the process of learning to use grounded rules successfully, both before and after the lesson.

Before the lesson, students vary widely in their prior knowledge that is relevant to these trigonometric identity problems. We considered whether this led to any differential effect between individuals. Students who lack the prerequisite attitudes, aptitudes or mathematical skills may have difficulty learning to the unit circle. In the extreme, this hypothesis seems trivially true, if we were to recruit infants as participants. Among the population of Stanford undergraduates, however, we did not find a clear interaction between block 1 performance and lesson condition on block 2 performance. A pilot study found that high school students performed near chance on block 1, and did not demonstrate mastery after the grounded lesson. Therefore, with the Trig Academy project, involving high school and community college students, we included a number of measures of prior knowledge and potential moderators. We also expanded the lesson into an interactive educational tool that students advance through over the course of multiple sessions. Students continue to participate in ongoing Trig Academy studies, and further work is required to establish relationships in the data that has been already collected.

After the grounded lesson of our Stanford studies, students reported using both rules and the unit circle. As they progressed through more trials, they reported relying less on the unit circle. Such self-reports (in isolation) might be subject to particular biases that also exhibit trends over the course of solving problems. However, in our Trig Academy studies, we recorded interactions with an external unit circle diagram. Consistent with the self-report finding, Trig Academy students used the external unit circle less as they solved more problems. Interestingly, there was a positive relationship between accuracy and students who tended to use the unit circle, but there was no relationship between use of the external unit circle and accuracy within students. Although additional investigation could help distinguish possible explanations, this suggests that students who are able and inclined to ground their reasoning in the unit circle achieve higher accuracy on average than students who do not externally show that ability or inclination.

Chapter 5

Conclusion

Mathematical cognition is a fascinating domain that recruits a wide range of cognitive representations. Trigonometric relationships are a powerful example of how a visuospatial, coherent conceptual structure can make symbolic expressions into more meaningful rules that can be applied and generalized successfully.

Chapter 2 introduced our empirical methods, and established the unit circle as a popular and successful model for solving trigonometry problems. In Study 1, we observed how students approach solving trigonometric identity problems, and found that students most often reported using the unit circle. Students who reported using the unit circle also tended to have higher accuracy. Other students, who did not report using the unit circle or reported using it less often, were more likely to rely on heuristics, like pulling out the minus sign. Study 2 randomly assigned students to a formal lesson or to a lesson grounded in the unit circle, and thus allowed us to test the causal nature of the relationship observed in Study 1. Consistent with Study 1, the grounded lesson facilitated problem solving.

In Chapter 3, we examined how students extend the benefits of the unit circle taught in the grounded lesson to novel transfer problems. Study 3 counterbalanced some problems as taught in the lesson or held out for transfer, and we analyzed specific patterns of learning and generalization over different types of problems. Study 3 also included a block of problems with problem-specific self-reported strategies, which constrained potential theories of how the grounded lesson facilitates transfer. Students reported relying slightly more on a rule or formula when solving taught problems than transfer problems. However, across all problem types, students often reported relying on both the unit circle and a rule or formula on a particular problem.

Chapter 4 explored the process of learning grounded rules, ranging from how students differ in readiness for the unit circle, to how students reduce active use of the unit circle as they gain confidence with grounded rules. While there was not a clear interaction between block 1 performance and lesson condition, we have been collecting a wide range of measures in our Trig Academy studies, which include attitudes, aptitudes, and mathematical skills. After their respective grounded lessons, students tend to report relying less on the unit circle, and indeed they interact less often with an external unit circle as they progress through problems.

Across our studies, grounding trigonometric relationships in the unit circle facilitated successful problem solving and generalization. I considered three primary hypotheses that could explain the benefit of the grounded lesson. Note that these hypotheses are not mutually exclusive. Rather, they are nested – each hypothesis is conditioned on some truth of its simpler predecessor. These hypotheses do, however, represent independent explanations, and I argue that each contributes to some extent to the success of the grounded lesson.

First, the grounded lesson may provide a procedure involving the unit circle that is sufficiently general for the problems (both taught and held out for transfer) in our studies. For instance, after looking at a probe expression, a student might map its symbols (or parts of the expression) to quantitative properties of the unit circle. The number(s) inside the parentheses of the expression can be represented as angles, rotations, or positions around the circle. The cosine or sine function is then interpreted as the projection of that angle's endpoint onto the x or y axis. A student can use the same unit circle approach to find an approximate quantitative value of each alternative, and can compare their magnitude and sign to value of the probe expression.

According to a second hypothesis, visuospatial reasoning can provide students with an effective standalone procedure for deriving answers to problems, but rules may have certain advantages. Having mastery of two different strategies may therefore help a student achieve both speed and accuracy. Indeed, students reported more use of rules after the formal lesson, while students who saw the grounded lesson reported more use of both rules and the unit circle. This suggests that rules do play an important role in problem solving, and that students were not simply using a unit circle-based quantitative interpretation.

Finally, while students may benefit from having rules as an additional independent strategy, our third hypothesis proposes that the unit circle and rules may be more intimately related, through mutual support or interaction. The grounded lesson could change the way students discover, use, and remember rules or rule-like representations to solve problems. For instance, the symbolic rule $\sin(x + 180) = -\sin(x)$ may be verbally encoded as "when an angle is rotated halfway around the circle, the y-coordinate of its endpoint has the same value but the opposite sign".

I have offered evidence that each of these mechanisms contributes to the success of the grounded lesson, but ardent believers and skeptics of each mechanism could offer critical interpretations of that same evidence, naturally suggesting further research efforts.

5.1 A Unit Circle Lesson Without Rules

First, a unit circle fan (or a rule skeptic) may argue that the unit circle is the key representation and underlies a successful general procedure. While I pointed to the increase in self-reported rule use after the grounded lesson as evidence of a more complex story, which involved rules along with the unit circle as parts of a successful strategy, the unit circle fan's potential responses include: (1) a student's mistaken interpretation (and therefore mistaken self-reported use) of "rule / formula" and (2) the lack of difference in accuracy between a student using rules and the unit circle and a student using the unit circle alone. This latter response motivated an experiment to distinguish these two types of strategies.

We conducted an intial attempt at such an experiment by using two conditions: the original lesson which grounded rules in the unit circle, and a lesson which focused on the unit cricle while reducing the role of rules. There were several ways in which the original lesson may have encouraged participants to adopt rules.

First, the grounded lesson included verbal generalized principles. These were originally written to match the grounded lesson as much as possible to the rule-only lesson in Study 2. For instance, on the $\sin(-50 + 0)$ page, students saw the following text, with italics added for emphasis:

> "Dismiss the zero because it does not affect the angle. Then imagine the angle which is formed by rotating 50 degrees clockwise or downward from 0, and examine the vertical component of this angle. Now compare this to the vertical component after rotating 50 degrees in the positive direction. The vertical component is the same distance from zero in both cases, but in opposite directions. More generally, the sine of a negative angle is equal to the negative of the sine of the corresponding positive angle."

This last sentence is identical to a sentence in the rule-only lesson, except with "angle" substituted for "value". The rule-only lesson also included algebraic formulae, but Koedinger & Nathan (2004) might argue that the verbal information is actually communicating the rule more effectively than the formula. Simply removing these sentences from the unit circle lesson and examining whether performance drops would show whether students were relying on these verbal principles.

Second, the lesson and the task itself may have emphasized problem types. The unit circle could serve a general procedure that works across problem types, whereas rules, such as those in the rule-based lesson, are often more narrow, being applied to particular problem types. An emphasis on problem types may have led to rule use. There are three ways to address this issue. First, the lesson featured examples that were selected to be one of each problem type (for each function, sine and cosine), and each example was featured on a different page. An alternate structure would emphasize the general procedure of using the unit circle to understand the value of an expression, and not have such apparent "types" of problems. Secondly, students may have already identified problem types in the pre-test stimuli. One test of this would be to ask students to generate stimuli after the pre-test. If students are indeed sensitive to the problem types present in the pre-test, we could eliminate the pre-test, and randomly assign students to lessons they see first, then administering a post-test. This would lose within-subject information, but would nevertheless be a valid between-subject design. Finally, both the rule-only lesson and grounded lesson referred to multiple techniques to learn for different problems. The rule-only lesson discussed multiple rules, while the grounded lesson discussed multiple relationships. We might wonder what students believe the lesson is supposed to be teaching, or what they believe they are supposed to be learning: a general model of mathematical meaning, or multiple narrow rules guided by the unit circle.

The nature of the multiple choice problems in the pre-test and post-test may also have pressures on different strategies. If students are using the unit circle strictly to find and compare the value of each expression, they must find the value of five different expressions (the complex probe expression, and four simple choice expressions). This extends the time required to use a unit circle approach, and may encourage students to find shortcut strategies. Fundamentally, the same questions could be asked using two alternatives (either positive vs negative, or sine vs cosine). For instance, given the expression $\cos(-50+0)$, your task is to choose whether it is equivalent to $\cos(50)$ or $-\cos(50)$. (In a previous study, participants chose one of these two responses on 87% of trials, across all strategies and levels of performance.) If we don't want to force a specific foil, just one simple expression could be shown on each trial (e.g., $-\cos(50)$), where the task is to determine whether it has the same or different value as the expression ($\cos(-50+0)$). This would reduce the number of expressions that a subject might evaluate. These problems would still allow rule use, but may reduce the potential cost of time in using the unit circle.

In light of the considerations above, we constructed a new lesson that excluded or modified these factors that pointed students towards use of rules, and we conducted a study comparing it to the original lesson. In this new lesson, we adjusted the general verbal principles to be simple statements of the particular case. For example, where we had previously said "More generally, the sine of a negative angle is equal to the negative of the sine of the corresponding positive angle" we now said "Thus, the sine of negative 50 degrees is equal to the negative of the sine of positive 50 degrees." We also removed verbal indications at the top of each page that may have designated different problem types. For example, where we had previously said "Consider this example, where zero is the special angle and the other angle is negative" we now simply said "Consider this example." Finally, we made minimal changes to the verbal framing of the lesson to suggest a general mathematical model of meaning, rather than multiple separate relationships. For example, where we had previously said "The relationships we have described above apply directly to some of the problems you saw in our first study, and will see again in the next part. The relationships are helpful with other problems as well, but may need to be adapted or extended to address all of the problems" we now said "The approach we have described above applies directly to some of the problems you saw in our first study, and will see again in the next part. The approach is helpful with other problems as well, but may need to be adapted or extended to address all of the problems."

The procedure of this new study differed from previous studies in a few ways, although we did keep the multiple choice structure of the test questions. First, we separated the pre-test from the lesson and post-test into two different sessions. This may have reduced any cues to the structure of the problems which could then facilitate identification of rules. We also did this separation to allow selection into the post-test. We believed the lesson would have maximal impact on those students who performed above chance but below ceiling on the pre-test. Finally, we collected data from Stanford undergraduates online, rather than in the laboratory.

After removing or addressing factors which might have encouraged students to use a rule-based approach, there were several different potential outcomes that we might have observed. If our new lesson increased unit circle use but did not increase rule use, and students nevertheless matched the performance of students after our original lesson, we may interpret the unit circle to be an effective standalone procedure for solving these identity problems. Alternatively, if students failed to reach the same level of performance as students who used the unit circle and rules in conjunction, we may focus more on how the unit circle facilitates comprehension and learning of formulae, rules, and heuristics. A final possibility is that successful use of the unit circle facilitates self-discovery and use of rules. If so, we would expect the original and new lesson to produce a similar increase in rule use and a similar increase in accuracy.

Consistent with this final possibility, we found both lessons increased self-reported use of rules, with no significant difference between the lessons in the magnitude of the increase. We also found no difference in overall accuracy, transfer accuracy, and response times, as shown in Table 5.1. The interpretation that the unit circle facilitates self-discovery and use of rules is unfortunately confounded with the possibility that we did not sufficiently reduce elements of the lesson that might suggest use of rules.

Table 5.1: Measures (with 95% CI) comparing the original grounded lesson with the new lesson reducing the role of rules, for each block and the change from block 1 to block 2. Independent t-tests compare the change in each measure between conditions.

Lesson condition	Block 1 *M*	Block 2 *M*	Change *M*	$t(79)$	p
Self-reported use of rules					
Original lesson	2.26 [1.93, 2.60]	2.74 [2.43, 3.02]	0.48 [0.05, 0.81]	0.22	.826
Reduced rules	2.54 [2.20, 2.83]	2.95 [2.54, 3.29]	0.41 [0.02, 0.78]		
Overall accuracy					
Original lesson	0.65 [0.59, 0.70]	0.72 [0.66, 0.78]	0.08 [0.03, 0.13]	0.07	.946
Reduced rules	0.63 [0.58, 0.68]	0.71 [0.64, 0.76]	0.08 [0.03, 0.12]		
Transfer accuracy					
Original lesson	0.61 [0.54, 0.66]	0.67 [0.59, 0.74]	0.06 [0.00, 0.13]	-0.12	.903
Reduced rules	0.59 [0.53, 0.65]	0.66 [0.58, 0.72]	0.07 [0.00, 0.12]		
Median response time (s)					
Original lesson	9.37 [8.09, 10.71]	8.48 [7.28, 10.39]	-0.89 [-2.03, 0.63]	0.78	.437
Reduced rules	9.55 [8.37, 10.89]	7.91 [6.74, 9.09]	-1.63 [-2.76, -0.12]		

5.2 Identifying Use of Grounded Rules

How we identify use of grounded rules remains an area for thought and future experimentation. A skeptic of grounded rules may question whether a student indeed used a rule grounded in the unit circle, as opposed to an ungrounded rule. In Chapter 3, I presented instances where students who reported using only rules actually expressed their strategy in terms of the unit circle. This evidence is however anecdotal (or simply an existence proof) and somewhat limited. There could be a number of students who report using only rules, and whose protocol does not contain obvious references to the conceptual structure of the unit circle, but who nevertheless did appreciate the relationship in the context of the unit circle. What kind of quantitative or systematic evidence could determine whether a student is using rules that are grounded? Suppose students had an external unit circle tool to manipulate during problem solving, and were subsequently asked to report their strategy. If a student reports using only rules, but actually interacts with the external unit circle during problem solving, it would certainly inform our interpretation of that student's self-report. The student could be under-reporting their circle use, perhaps considering it a back-up strategy or confirmation of their answer. I would argue that it is plausible these students consider their rules to be grounded in terms of the unit circle, and so while they did not feel they were relying on the unit circle as a strategy, they did rely on its structure to appreciate the relationship.

While I have discussed the hypothetical use of the unit circle as a standalone general procedure, it may in fact be difficult or rare for students to do so without relying on rule-like knowledge of general relationships. I believe that there is some contribution of the unit circle independent of rules, although the adoption of rules (and grounded rules in particular) make quantifying this contribution difficult. As a spatial model, the unit circle (when static and external) represents quantities as specific, determinate instances. (Note that the unit circle may have greater representational power as a dynamic or internal mental representation.) However, the angle instantiated in the unit circle model is not necessarily the same angle described in the expression of a problem. To test whether the correct instantiation matters, we could instruct students to save time by ignoring the angle in a particular expression and instead visualizing a small fixed angle for every problem. As a manipulation check, we could explicitly measure the recall or recognition of angles, or perhaps better, we could use some implicit measure, where the angle can prime and facilitate a basic number task, or a spatial location task. The unit circle procedure and rule-based approaches are both mathematically valid strategies across all angles, but that validity is explicit and necessary in rules, and more implicit and emergent for the unit circle. This is closely related to the instantiation of angles in the unit circle. In the problem-specific report block of Study 3, we included problems with the generic variable θ, but these kind of problems require further investigation. To test how the unit circle users extend the validity of their model, we could also replace the value of the angle with more challenging items. For instance, we could have problems involving 45 degrees, which results in contingent equivalence that does not extend to other angles. Or we could use very large angles that are tough to simplify (such as 2^{100} degrees) or perhaps imaginary numbers.

Further pilot testing is needed to settle on the best experimental procedure for identifying use of grounded rules and disentangling it from other strategies. The problem-specific self-reports in Study 3 were a step in this direction, helping us to understand in more detail the strategies individual participants were using on particular problems.

5.3 Conclusion

Across our studies, the unit circle revealed itself to be a strong, coherent conceptual structure, and it creates a fertile ground for learning and understanding trigonometry through the interplay of visuospatial and rule-based approaches. While the unit circle could serve as a standalone procedure for solving problems, and it could also be used alongside independent rules or formulae, I have argued that successful students learn and use rules that are more intrinsically tied to the unit circle. Students learn how to map parts of a symbolic expression onto meaningful properties of the unit circle, and this grounded understanding facilitates application and generalization of rules in order to solve problems successfully. While I have focused on trigonometric identities, we should acknowledge that some topics in trigonometry, or higher-level mathematical reasoning more broadly, may be more amenable to reliance on visuospatial representations than others. Research across a range of mathematical topics is needed for a full understanding of the use and interdependence of different representations. Likewise, different individuals may have different abilities, and certainly have different proclivities, which may influence the success of a grounded lesson. Shedding light on these issues will have important implications, both for our understanding of the role of grounding symbolic expressions in visuospatial representations, and also for helping students learn trigonometry more effectively.

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